THE MATHEMATICAL GAZETTE

EDITED FOR THE MATHEMATICAL ASSOCIATION BY

R. L. GOODSTEIN

WITH THE ASSISTANCE OF

H. M. CUNDY K. M. SOWDEN

FEBRUARY 1957

Vol. XLI No. 335

"Teaching the Teacher." G. L. Parsons *page* 1

Universal Rectilinear Dials. A. W. Fuller 9

Pythagorean Triads in Babylonian Mathematics. E. M. Bruins 25

The Decision Problem. R. L. Goodstein 29

Archimedean Screws. H. G. ApSimon 38

Mathematical Notes (2661-2676). A. J. Carr ; J. Crampin ; W. J. Fearns ; W. Hope-Jones ; F. G. Maunsell ; D. S. Mitrinovitch ; E. H. Neville ; T. H. O'Beirne ; E. J. F. Primrose ; W. W. Sawyer ; A. G. Sillitto ; B. Thwaites ; H. W. Unthank ; G. N. Watson ; A. E. Western. 41

Reviews. D. W. Babbage ; T. A. A. Broadbent ; R. A. Brooker ; B. C. Brookes ; M. P. Drazin ; H. O. Foulkes ; A. Fröhlich ; R. L. Goodstein ; E. A. Guggenheim ; A. J. MacIntyre ; C. G. Paradine ; E. J. F. Primrose ; W. W. Rogosinski ; H. S. Ruse ; D. E. Rutherford ; D. B. Sumner ; G. Temple ; S. Vajda. 61

Gleanings Far and Near (1872-1876) 24

Address of the Mathematical Association, the Library, and of the Hon. Treasurer and Secretaries. 80

Conference for Teachers of Mathematics in Schools i

5s. 6d. net

G. BELL AND SONS LTD

PORTUGAL STREET · LONDON · W.C.2

G. BELL & SONS, LTD., PORTUGAL STREET, LONDON, W.C.2

ELEMENTARY ANALYSIS

by A. DAKIN, M.A., B.Sc., *and* R. I. PORTER, M.A. 10*s.* 6*d.*

This standard book, now in its 14th Edition, provides a very suitable year's work for the first year in the VIth form. " Very lucid and well-arranged, and the numerous examples provide excellent practice for both average and more advanced pupils." THE A.M.A.

FURTHER ELEMENTARY ANALYSIS

by R. I. PORTER, M.A. *5th Edition. Demy 8vo.* 21*s. net*

Completes a two-year Sixth Form course in Pure Mathematics suitable for all but the mathematical specialist. A SENIOR MATHEMATICS MASTER writes: " I have been teaching now for 40 years. All that time I have been looking for a good Analysis book. At last, I think, I have found it. Please pass on my congratulations and thanks to Mr. Porter for giving us such a fine textbook."

SHORTER ADVANCED TRIGONOMETRY

by C. V. DURELL, M.A., *and* A. ROBSON, M.A. 7*s.* 6*d.*

An abridged form of the authors' standard textbook *Advanced Trigonometry*, sufficient for the needs of many Sixth-Form students. " From experience of the earlier textbook with specialist pupils, we can unreservedly recommend the present issue." SCOTTISH EDUCATIONAL JOURNAL.

HIGHER CERTIFICATE CALCULUS

by C. V. DURELL, M.A., *and* A. ROBSON, M.A. 5*s.* 6*d.*

A selection from the material of Volume II of the authors' *Elementary Calculus.* " Principles are explained clearly and concisely, typical examples are worked out.... An excellent book." SCOTTISH EDUCATIONAL JOURNAL.

G. BELL & SONS, LTD., PORTUGAL STREET, LONDON, W.C.2

Photo by Humphrey Saunders

G. L. PARSONS
President, 1955–56

THE

MATHEMATICAL GAZETTE

EDITED BY

PROF. R. L. GOODSTEIN, UNIVERSITY COLLEGE OF LEICESTER

WITH THE ASSISTANCE OF

DR. H. MARTYN CUNDY, THE BEECHES, OBORNE ROAD, SHERBORNE, DORSET

MISS K. M. SOWDEN, NEWTON PARK COLLEGE, NEWTON ST. LOE, BATH

VOL. XLI FEBRUARY, 1957 NO. 335

"TEACHING THE TEACHER"*

BY G. L. PARSONS

I wish to bring to your remembrance some of the main general ideas with regard to the presentation of Mathematics which have from time to time been brought before the members of the Association. Articles of belief are notoriously difficult to isolate and those of this society are no exception to the rule. Like the classical theological creeds the ideas they present have often been current long before they are postulated dogmatically. Many of these ideas, in fact, seem to appear as a sort of common sub-conscious agreement, rather than to be postulated at all. The analogy with the classical creeds can be carried a little further, for, like those celebrated documents, the articles of belief of the Association have tended to emerge either as a response to some demand of the times or as a counter-blast to some heresy. They are often, in the first place, not *ex cathedra* pronouncements, but attitudes of mind (" climates of opinion " I believe is the modern jargon), arising out of the experience of individual teachers, who become aware of some particular need and are interested in the development of the measures necessary to supply it. Further, too, the validity and importance of certain articles of belief seems to alter with the passing of time, so that what appears vital to one generation assumes less importance in the next and may later be almost entirely superseded by newer ideas and interpretations. In the 85 years' life of this Association there has been plenty of time for this factor to operate and if the Association is to retain the life and vitality it has shown through those years, it is inevitable that other changes of a similar kind will take place. Nevertheless I believe that there are certain basic ideas which have been continually brought before our members and that it may be fruitful to give some thought to these and to attempt to analyse their strength and their weakness.

First and foremost among the ideas promulgated by the Association was undoubtedly the need for a new outlook on the teaching of Geometry and for something like the first 30 years of the society's life this provided the sole reason for its existence. Now that this objective has, to some extent, been attained it is perhaps right to look back to find out exactly what has been achieved and what has been lost. When we speak nowadays of Elementary

* Presidential Address to the Mathematical Association April 1956.

Geometrical teaching, we are apt to think of topics and fundamental theorems and the mode and order of their presentation and the very fact that we can talk in this way and be understood is a proof that earlier battles have been fought and won. Those who are slightly older may also think from time to time of those earlier controversial points so ably dealt with in the 1921 First Report, such as the Parallel Postulate, Superposition, Symmetry and the like. But it seems to me that this too, important as it was, was to some extent a later stage and that the first point which the small group of enthusiasts who formed the driving force of the Association in the early years wished to put before their fellow teachers was something much more fundamental. What they felt, I believe, was that the attitude to the study of Geometry (or as they would have put it, of Euclid) which they found around themselves was a relic of the mediaeval attitude which saw Geometry as one of the seven liberal arts, a compendium of the learning of the past, a smattering of which, in company with Grammar, Rhetoric, Logic and the rest, ought to form part of the training of every educated man. Such a study must of necessity become ossified and traditionalised and those early pioneers saw quite clearly that such a dead-letter pursuit could not lead to live results and that something more vital must be put in its place. I do not suppose that any of us will want to disagree with this laudable objective or to deny that it has, to some extent, been achieved. Nor need I take up your time with a discussion of the steps which have led to the abandonment of Euclid's attempt to start with a completely logical basis and to the substitution for this pseudo-logical basis of a set of intuitive ideas on Parallelism, Congruency and the like, for these too have, except in the most backward circles, found complete acceptance. But when I ask myself whether the final objective of something living, vital and developing has yet been reached I must confess that I am not so sure of the answer.

There are however, three or four other ideas of a more detailed character arising out of the reform of Geometrical teaching which have from time to time been urged upon the members of this Association and which seem to merit some comment.

At quite an early stage the suggestion was made that, after a short preliminary course, the main part of Geometrical instruction should be divided into two parallel courses, one consisting of formal theorems and the other of constructions. The first part of this idea—amplified by later reports into the well-known Stage A—has secured almost universal acceptance but the idea of parallel systems of formal and construction work, in spite of the appearance of at least one excellent text-book written on these lines, proved to be of less value than was expected. This was partly due to the fact that the border-line between Practical and Theoretical Geometry is too difficult to define and partly to the fact that the newer idea of a Stage B Geometry, deductive in character, but consisting of the study of groups of cognate geometrical properties (including relevant constructions) superseded the earlier suggestion. This idea of parallel practical and theoretical courses, though superseded, was nevertheless productive of much valuable material, since it served to focus attention continually on the application of geometry to non-theoretical situations of all kinds. This consideration, in spite of the fact that I have heard it regretted by geometers of the first order, is nevertheless valuable for lesser minds.

A second way of thought, which like the one I have just mentioned, has its origin in the 1902 Report on the Teaching of Elementary Mathematics—and I would like to remind you that we still number among my honoured predecessors the Secretary of the Committee which produced this report—was the idea that the real test as to whether the pupil is deriving anything from his study of Geometry lies in the extent of his power to apply his knowledge to new situations. This emphasis on rider work is superficially a very attractive

idea and has, like the other ideas considered, led to much that is valuable. I should be the last to advocate that the attempt to make pupils apply their geometrical knowledge to the solution of riders should be abandoned. But it is none the less true that the power to make this kind of application does not seem to rest only on a knowledge of geometrical facts but also on some indefinable quality which enables some people to solve jig-saw puzzles easily while others cannot. It may also be true that the greatly increased numbers of the school population has some bearing on this matter, since all the earlier doctrine of the Association was based on experience derived either from the Public Schools or the larger and more progressive Secondary Schools. Be that as it may, it is my opinion that one of the things the Association has not always "taught the teacher" with sufficient intensity is that some of his pupils will always find this particular exercise difficult.

This emphasis on rider work has led in some quarters to a development which I personally feel to be mistaken viz. the demand for the abolition of all set theorem work. Expert opinion differs widely on this matter and it must not be assumed that in voicing my own opinion I am in any way stating the policy of the Association. But it is a topic which I cannot honestly pass over for I believe that it is not possible to know any geometry at all worth the name unless it is based on a logically coherent set of facts, that is on a body of theorems (I mean by logically coherent no more than the words imply—it is the link between the various items which is to be logical—not the starting-points). I also believe that a very good way to fix these facts in the mind of the pupil is to demand that he should be able to reproduce the steps by which they are arrived at. This process has another advantage which I have heard quoted, namely that it provides the pupil with some valuable patterns of the various kinds of geometrical proof. It may very well be that exceptionally able pupils, and perhaps exceptionally able teachers too, may be able to dispense with this discipline but we must not cater for a minority of either kind.

Consideration of the corpus of geometrical facts leads naturally to the question of the sequence and order in which such facts are to be presented. All revolutions lead to a period of chaos in which all sorts of ideas about new freedoms and the breaking of old bondages naturally come to the fore. Perhaps the chaotic period resulting from the revolution in the teaching of geometry was to some extent prolonged by the intervention of two wars but it may not be out place to express the hope that it may soon be ended.

The only alteration of sequence recommended by the 1902 Report was the postponement of Euclid Book II till after the first part of Book III. The practical result of this was to bring in a change which has become almost a permanent feature—the postponement of the "extensions of Pythagoras" till after the early sections of Circle Geometry. The other major suggestion which has been put forward—the early introduction of Similarity—was a later product and has secured only partial acceptance. In its most extreme form, which consists of the replacement of the fundamental concept of Congruence by a fundamental concept of Similarity of which Congruence is a special case, it does not seem to have enlisted the support of many teachers and, if I may once more venture upon a personal opinion, I do not think that it will. The intuitive concepts of Similarity of Figures may seem deceptively easy to the adult mind already possessing some developed geometrical sense, but in my experience the reduction of these intuitive concepts to a mathematical form is not an easy matter and the process does not seem to me suitable for too early a stage. Perhaps I am expressing a very reactionary point of view. but one of the things I seem to have picked up in the course of 35 years' teaching is that old Euclid was rather a wise schoolmaster after all.

Freedom of experiment is an excellent thing—though the fact that the subjects upon whom the experiments are to be performed are human beings

with lives to live and ambitions to be achieved must not be forgotten. It is an excellent thing that we have retained an educational atmosphere in which people who try out new ideas are not burnt at the stake as heretics. But those of us who from time to time have to concern ourselves with the examining of candidates in Geometry may be forgiven for wondering whether this freedom has not gone far enough and whether the time is not ripe for some degree of consolidation. It is not perhaps irrelevant that when I was casting about for a subject for to-day's remarks, one of your most prominent and well-respected members (by no stretch of imagination a diehard) suggested as a title " Shall we have to give up examining Geometry? " and it is interesting to note that this query was foretold at the Annual Meeting 51 years ago. Now I am afraid that I have little sympathy for the oft-repeated suggestion that examinations do not matter. We may perchance deprecate their undue influence: we may even be bold enough to think that, given a free hand, we ourselves could devise a much better method of assessment: but it is of little use to bury our heads in the sands and pretend that something which is an important part of our educational system does not exist. If this is so, and if we value the retention of geometry as an educational instrument, not merely for the gifted few, but for the many, as I believe we do, than we ought to see that our efforts to remould the teaching of the subject do not take a shape which leads to its ultimate destruction—a result which will inevitably follow if it becomes impossible to test it. I have no specially detailed suggestions to put forward with regard to the vexed questions of geometrical sequence and even if I had, a Presidential address would be no place for them. But I do suggest that there is much food for thought in these matters.

There is another important item of geometrical advice which the Association has consistently put before its members—namely the desirability of giving our elementary geometry a much more definitely three-dimensional character. If I pass over this item briefly I trust that I shall not be thought to be minimising its importance. It is a piece of advice we all ardently desire to follow, at any rate in our more conscientious moments, and I am sure that we all find it no easy task. But as I notice that this very topic is to be discussed at a later session, I shall content myself with observing that I look forward to seeing the Association once more at work on its task of " Teaching the Teacher " in this important respect.

In concluding our review of the ideas which the Association has tried to put before its members in the elementary geometrical field, we may therefore say, I suggest, that these ideas have been productive of many valuable results but that there are still important problems to be solved.

I pass now to the consideration of another general idea of great moment which has formed the background of much of the Associations's work. I refer to the concept of the " unity-in-variety " of mathematics. It is difficult to trace the exact origin of this idea: it received some mention in Prof. Perry's famous address in 1901 and it was also, no doubt, in the minds of the members of the A.I.G.T. who decided, round about the same time, that the object of their society should be extended so as " to bring within its purview all branches of elementary mathematics ", a form of words which still appears on the official leaflet of the Association. There are several ways in which this basic concept has made itself felt. The first, and perhaps the most outstanding, of these is that Elementary Mathematics—that is to say the part of the subject studied by *all* secondary school pupils—ought to consist of more than the traditional Arithmetic, Algebra and Geometry. Hence we meet, even so early as the 1902 Report, the suggestion that mechanical skill shall be subordinated to understanding and that parts of the then current syllabus for Arithmetic and Algebra might be pruned to allow time for the introduction of Graphs. This was followed, at a later date, and not without controversy, by the intro-

duction of numerical Trigonometry and, as you know, at present the frontier is in a somewhat fluid state with regard to the admissibility of elements which may be described, according to taste, as functionality, analysis or even elementary calculus.

This idea has evidently been productive of much success, as anyone who cares to compare examination papers, whether public or private, of 1905 or thereabouts with those of to-day can easily discover. Many of us have experienced in the course of our own teaching career the change of status of Trigonometry from a subject studied only by the few to an experience which is shared by nearly every grammar-school boy or girl. I use the word "experience" deliberately for many of us have noted with pleasure the avidity with which a new piece of knowledge of this kind has been welcomed by pupils whose skill at the traditional parts of the elementary mathematical syllabus has been poor and who have become bored at the well-meant efforts of their teachers to improve it. More recently, too, the demand for a closer integration between Geometry and Trigonometry has greatly lightened the load at one or two points.

But there are certain dangers and I should not be dealing with the situation honestly if I ignored them. Mathematics owes its privileged position in the educational curriculum to the belief entertained, rightly or wrongly, in the world outside the classroom that a knowledge of its elements (vouched for, maybe, by a G.C.E. pass at O level) is in some mysterious way a guarantee of proficiency in those processes of reckoning, counting, measuring, weighing and forecasting which play a part in commercial life. As we spread the content of our courses in new directions we inevitably decrease their intensity. Like any other industry we do depend to some extent on the quality of our end-product. This quality includes, of course, much more than the ability to count, measure, etc.—it implies, for example, those characteristics of perseverance, intellectual honesty and truthfulness which I am old-fashioned enough to believe the study of mathematics tends to inculcate—but the fact does remain that the first check made on the value of our end-product is to discover whether it can add, subtract and multiply correctly, and if this test is not passed the tester is apt to assume (quite unwarrantably of course but none the less obstinately) that subsequent tests in those other qualities to which I have referred may well end in similar failure. It is easy to laugh at such ideas, to brand them as Philistine and cultureless, but I often wonder whether some of the fault is not ours as teachers and whether in our zeal to open the gates of higher learning we do not sometimes yield to the temptation of accepting too low a standard of performance in straightforward things. No doubt I shall incur your displeasure by my too-frequent references to examinations but to many of us who have been engaged in the conduct of these unpopular events for many years the fall in the standard of elementary computation presents a challenge which we do not particularly like and the continual appearance of a number of candidates who can quote difficult Trigonometrical formulae, can differentiate, integrate, etc., but who make elementary mistakes in the simplest steps of Arithmetic and Algebra, at any rate sounds a warning that the process of extension of the syllabus has dangers which need to be closely watched.

Two other ideas which have emerged from this concept of the unity of mathematics with which we are dealing are cognate. One is that some attempt should be made to present the subject in its historical development. On this I need only make a short comment. I am quite convinced that this should not mean, at least as far as the non-specialist is concerned, a course in mathematical history but that the idea should figure as a continual background behind the presentation of the topics studied. Some topics are naturally better suited to this than others—the process of completing the square is one of the most obvious examples. But as the Teaching Committee is at present

working on this point further comment to-day is unnecessary. The other cognate idea is the conception that the purpose of learning mathematics is not merely the acquisition of certain routine skills—though as I have said above I do not consider this as useless—but that beyond all this there should grow in the mind of the pupil an awareness that he or she is making the acquaintance in a limited way of one of the most powerful and fascinating modes of human thought. This, like the historical idea, rests to a large extent on the attitude of the teacher, for if he treats the whole thing as a boring routine which has to be endured his attitude will soon communicate itself to the pupil and that will be the end of any wider conception of the subject. Of many occasions when some explanation of the nature of mathematical thinking is appropriate, two come to mind : viz. the introduction of the trigonometrical ratios of obtuse angles and the early steps in negative and fractional indices. For in each of these topics one of the characteristic modes of mathematical thought, ever seeking to include new categories without excluding or contradicting the old, shows itself clearly. Perhaps also, even at the Fifth Form stage a pupil can be asked to consider for a moment how his own idea of what he means by a number has developed through the years.

It is no part of the Association's service to its members to speak of text-books, except in the reviews of the Gazette, but a discussion of the ideas behind the " unified course ", the fourth and last of the applications of the " unity-in-variety " principle, naturally suggests an allusion to the new style of comprehensive text-book. In the presence of so many distinguished authors it would be both rash and invidious to attempt any authoritative pronouncement on such a matter. Nevertheless there is one general remark which seems relevant. The conception of the unity of mathematics about which I have been speaking must first of all exist in the mind of the teacher, who must seek a method for making it come alive in the mind of the pupil. Whilst a well-constructed text-book can undoubtedly assist in this process, it cannot by itself perform it and if the concept is not sufficiently alive in the mind of the teacher, or if he does not think it to be of sufficient importance, no text-book and no re-shuffling of the content of the syllabus can possibly produce a worth-while result. Experiment and new presentation are essential for progress and as we think of this it is fitting to pay a passing tribute to our mathematical publishers who, during these times of change and experiment have shown a breadth of vision and a sympathy with new ideas which have often far transcended the sordid limits of profit and loss.

There has been for many years a cry that mathematics must be made practical. This too is an ideal which the Association has ever been ready to put before its members. I need not weary you with a series of references to those excellent reports on the teaching of various branches of the subject where detailed suggestions along these lines are to be found. It is due to these ideas, which again go back as far as 1902, that many topics which could be found in the syllabuses and papers of that time have disappeared, that the complicated manipulations of algebraical expressions which were the bugbear of our youth have been modified and eased, that our financial transactions have passed, so to speak, from Lombard Street to suburbia and that our arithmetic concerns itself with what seem, to teachers at any rate, everyday topics. What is disconcerting is that this air of practicality does not always communicate itself to the pupil, as witness a recent problem in which a horse, which was said to be engaged in the eminently practical task of rolling the school cricket-field, was calculated by not a few pupils to have walked some ten times round the circumference of the earth in the process. I sometimes wonder whether we have not gone so far towards removing the so-called drudgery that we haven't removed the accuracy as well. I am in any case, a little suspicious of the word " drudgery ", which seems to me to belong to the

vocabulary of the middle-aged landlady rather than to that of the starry-eyed bride. Or to change the metaphor, are we, in our attempts to bring everything down to the level of the pupil, in danger of substituting for the eager climbing of the ladder of knowledge a bored journey on a dull escalator? There is one part of our work where I am pretty sure that this is so. It is the thing, we are told, to give our pupils a thorough training in the use of slide-rules, tables and all sorts of mechanical devices. Engineers, they say, always have these at their elbow and it is the only sensible way to go on. I cannot find myself in entire agreement for in my view such aids are a very good thing for those who can perfectly well do without them. But there is a great difference between saving time in the performance of something you know how to do well and saving yourself the bother of trying to do it for yourself. When I see boys performing such operations as dividing by 4 by logarthims, or using the sine-rule to calculate the sine of an angle in a right angled triangle—not to mention that prince of practical mathematicians who used his log tables to find the square root of unity (he found it to be ·003162)—when I see such things happening not merely occasionally, but with some regularity, I may be pardoned for wondering whether we do not sometimes carry our zeal for doing things the way the practical man does them a little too far.

This search for the practical has not, of course, confined itself to simple matters like Arithmetic. It has run through the whole range of mathematical teaching and in one particular respect has produced a profound modification by the introduction of what is sometimes called "Mathematics for Science". The idea that there is one sort of higher mathematics suitable for schoolboys who propose to study Mathematics at the University, and another sort suitable for those whose interests lie in the field of Physics, Chemistry and Engineering is a significant development of this century and its full meaning has yet to be worked out. The change of interest from the field of Geometry to the field of Analysis which has been a marked feature of recent years may not be entirely due to this desire for the practical, but it has certainly something to do with it.

Some at least of older teachers like myself are inclined, in part, to deplore this change and the consequent eclipse of many of the elegant methods of Geometry, whether Analytical or Projective, on which we were brought up. I may be mistaken, but it has always seemed to me that this Association has stood out against the application of the "Mathematics the handmaid of Science" thesis, at any rate in its crudest form.

There is another idea which the Association has given to its members, to which I must now refer, namely the idea of "stages of learning". This is of particular interest because I feel that all its implications have not yet been worked out. The idea of progressive stages of presentation of mathematical material, which we have now learned to call Stages A. B and C, stems I believe from the First Report on the Teaching of Geometry, which was the first publication of the Association to use this nomenclature. What is interesting is the fact that we are only just beginning to discover that a similar discrimination between the stages of learning is valid in other branches. Prominent among the parts of the subject to which this type of approach may be applied are Trigonometry, Analytical Geometry and Calculus. One of the reasons why I have mentioned this topic is that it seems to me that our experience with regard to Geometry is beginning to repeat itself with regard to these other branches, particularly, perhaps with regard to Calculus.

The Second Geometry Report makes it clear that while the content and application of Stages A and B have, during the course of the years become more or less common property, the working out of Stage C has not followed suit. I do not wish to discuss this state of affairs with regard to Geometry: possibly the swing away from purely geometrical studies which has happened during the past few years may have rendered what was originally envisaged

as Stage C Geometry (the examination of the fundamental postulates) unnecessary or out-of-date. What I do wish to point out is that a similar state of affairs seems to be arising with regard to Calculus and Analysis. Here again we have a fairly clear idea of what Stage A implies. It is that part of the work which is often covered by good pupils at the Fifth Form stage probably in preparation for Additional Mathematics at O level. It usually consists of a fairly complete review of the methods of Calculus, but with applications only to a restricted class of functions such as polynomials and, in some cases, elementary trigonometrical functions. Stage B of the work is also beginning to be clarified. We are perhaps not yet quite sure what should be restricted to the purely mathematical specialist in the Sixth Form and what should be the common property of those who study mathematics for its own sake and those who study it principally as an instrument in scientific work. In one well-known syllabus, for example, the differentiation of inverse trigonometrical functions is excluded from the requirements for scientists, while the associated integrals seem to be required as a part of the study of Simple Harmonic Motion. But in general much good progress has been made in these directions and we seem to have agreed that certain things (e.g. the Theorem of the Mean) are not proper parts of Sixth Form Calculus, which may be imagined to be more or less restricted to Stage B. It is not yet at all clear what is the desirable content of the Calculus Stage C and where and when it is to be attempted, and what is true of Calculus is also true to some extent in Analytical Geometry and elsewhere. If I may venture an opinion, it seems to me that the liaison between the various stages of teaching is weakest at the point where the Sixth form mathematical student merges into the first year undergraduate and that there is at least a danger that in regard to such matters as I have quoted each party is expecting the other to do the work and in the end it may well happen that it is well done by neither. The situation is, of course, temporarily aggravated by the difficulties attendant upon National Service requirements, and these make it all the more necessary that we should have clear ideas on these matters.

In recent years a number of topics which used to figure in the training of a potential mathematical scholar have tended to drop out of the school syllabus but somehow or other the student is still expected to know them by the end of his first year at the University and the question of when and how he is to learn them seems to me to need some investigation. Perhaps this is the right moment to put in a plea for the continuance of that happy liaison between dons and schoolmasters which has been such a beneficial feature of the work of this Association. This is possibly a matter for the younger generation of university teachers to take to heart and, whilst acknowledging the many calls on their time in these crowded days, we do hope that some of them will find time, as their older colleagues have so willingly done in the past, to discuss mutual problems with us and take an active part in our work.

The fact that since these remarks were first drafted a recent conference has urged upon all concerned the necessity of a stringent review of Sixth Form and university requirements with a view to reducing the load laid upon the student lends further point to what I have just said, for I sincerely hope that no such review of mathematical requirements will take place without the opinion of this Association being sought upon the matters concerned in the review.

I am well aware that this short review of the ideas with which the Association has attempted to teach this particular teacher is a personal one, and I daresay some of the Association's teaching, like much of mine and yours, will have been misunderstood. We all may take encouragement from the fact that the process can still continue and that the conclusion of this address will set you free to listen to the remaining papers from which we may all hope to learn much.

G.L.P.

UNIVERSAL RECTILINEAR DIALS

By A. W. Fuller

I first came across these dials at the Old Ashmolean Museum when looking at astrolabes. Miss Kathleen Higgins, then writing a thesis on dials, asked if I could explain the operation of a dial described by Regiomontanus (1436–1476) of which several examples are shown in British and European museums. That was in 1950 ; since then I have repeatedly tried to evolve an explanation of some way in which dials of this kind may have been invented. Only recently have I been satisfied with my results.

Regiomontanus's Dial: Figure 1

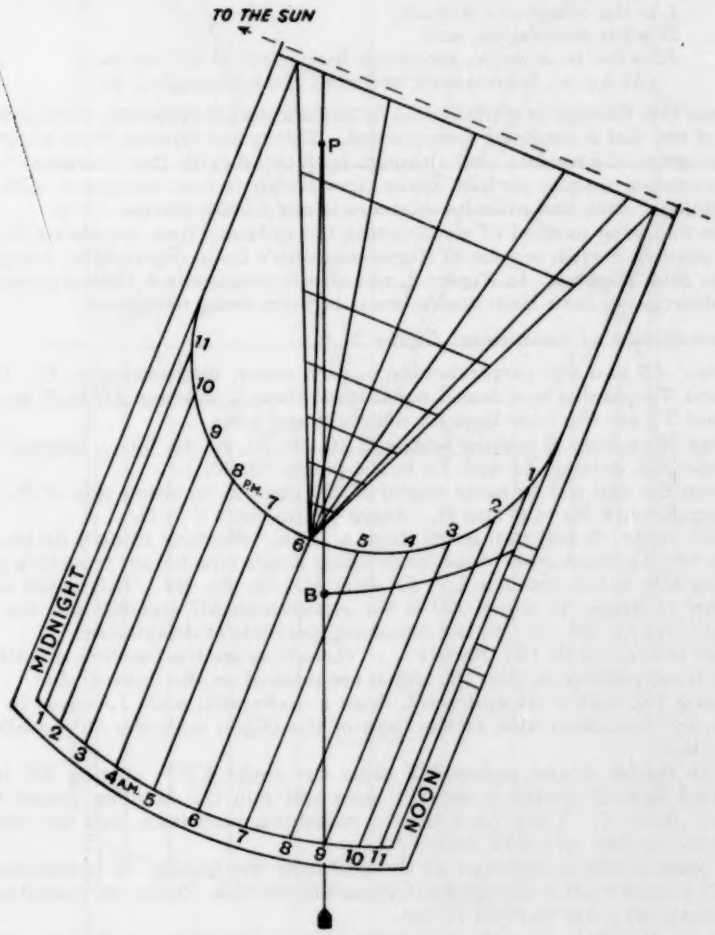


Fig. 1.

Generally this dial consists of a plate smaller than hand size with holes for aligning the upper edge with the sun's rays. The inclination of this edge is then equal to the sun's altitude. A string with plumb bob hangs from an articulated arm in such a manner that the point of suspension P can be moved over the face of the dial and set in a position depending on latitude and declination, these two settings being independent. A bead B is then moved along the string to a position depending on the declination. The dial being thus set up according to place and day, its edge is directed towards the sun, and the time of day is estimated according to the position of the bead, hanging between straight lines calibrated in hours.

The formula on which the operation of the dial depends is a particular case of the Cosine Rule in Spherical Trigononometry :

$$\sin A = \cos L \cos D \cos H + \sin L \sin D\ ; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where A is the sun's altitude,
L is the observer's latitude,
D is his declination, and
H is the hour angle, measured from noon at 15° an hour.
(At 4 p.m., for instance, or 8 a.m., the hour angle is 60°.)

Since this formula is symmetrical in latitude and declination, the construction of the dial is similarly symmetrical. This is not obvious from a cursory examination of a model ; and although both types exist, the " reversed " dial is uncommon despite certain advantages which it has, compared with the " ordinary " dial, and even its existence is not widely known.

The following method of constructing the ordinary dial, translated from a 17th century French version of Regiomontanus's Latin description, was given me by Miss Higgins. In Figure 2, not all the construction lines required by this description have been drawn, some of them being redundant.

Regiomontanus's Construction: Figure 2

Draw AB and CD perpendicular to each other, intersecting in E. Draw Xx and Yy parallel to CD and equidistant from it, cutting AB in X and Y. Xx and Yy are the hour-lines for midnight and noon.

Draw lines from E making angles of 10, 20, 30, 40, 45, 50, ... degrees with EA and EB, cutting Xx and Yy in the points 10, 20, ...

Draw EG and EH to make angles of 23½ degrees on either side of EC and cutting Xx and Yy in G and H. Draw GH to cut EC in O.

With centre O and radius OG draw a circle. Starting from G divide this circle into 12 equal arcs. Join the dividing points equidistant from G in pairs cutting GH in the points a, b, c, d. Join Ea, Eb, Ec, Ed. This figure is the TRIGON of Signs, in which EO is the equinoctial, EG and EH are the two tropics and Ea, Eb, Ec, Ed the remaining parallels of declination.

Join the points 10, 10 ; 20, 20 ; ... . These lines are the parallels of latitude. Only those portions within the trigon are marked on the finished dial.

Along Yy, with Y as mid-point, draw a declination scale IJ equal to that made by the intersection of the lines of the trigon with the 45° parallel of latitude.

With centre E and radius EX draw the circle XFY, cutting EC in F. Starting from X divide it into 15° arcs and join the dividing points equidistant from X. These lines are the remaining hour-lines, and are usually bounded by two arcs with centre F.

A jointed arm is attached to the dial near the trigon. A plumb-line on which a small bead is threaded is fastened to the tip. Sights are placed above the trigon on a line parallel to GH.

To use the dial, place the tip of the arm on the trigon at the intersection of the appropriate lines of declination and latitude. The plumb-line is laid

across the declination scale IJ at the correct point and the bead moved along until it lies on the scale (i.e., on the noon hour-line). The dial is then held vertically with the sights directed towards the sun ; the position of the bead gives the time.

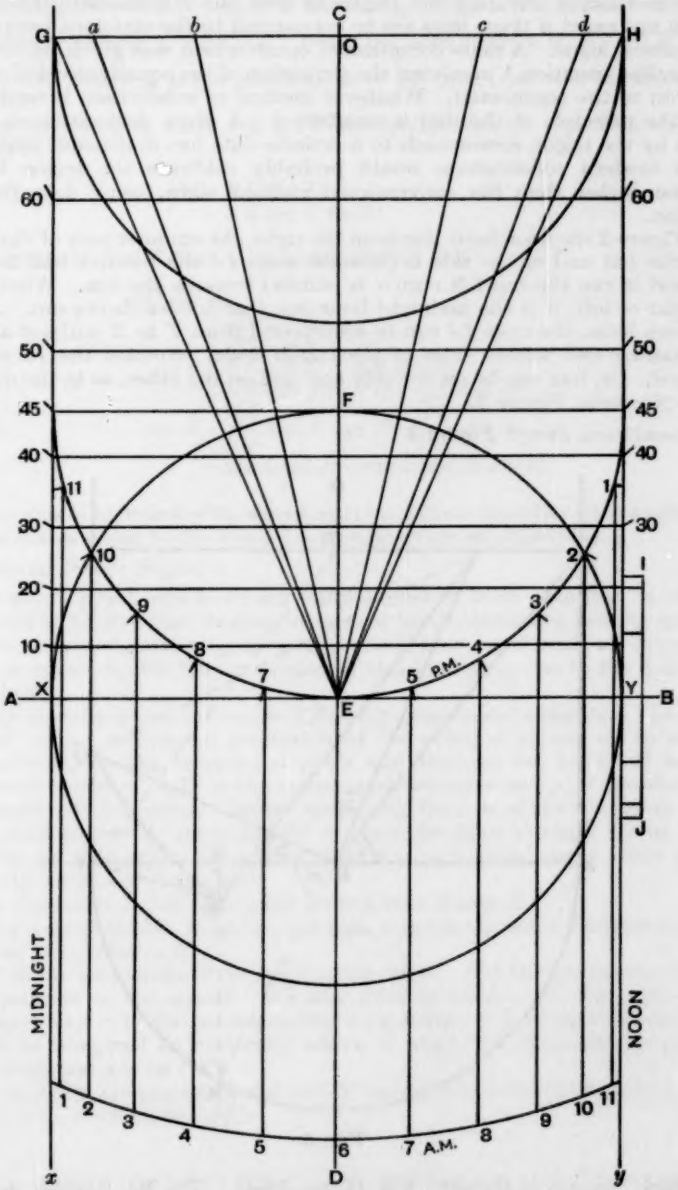


FIG. 2.

This construction calls for few comments, but some may clear up difficulties.

It is not necessary to draw all of the construction circles: one circle, suitably divided, will suffice if parallel rulers are available, or if drawing-board techniques are used.

The method of dividing the trigon to give the intermediate declination lines is not exact if these lines are to correspond to the standard positions of the zodiacal signs. A more complicated construction was given by Gunther in a similar situation,* involving the projection of the equally-divided ecliptic circle on to the equinoctial. Whatever method of subdivision is used, however, the principle of the dial is unaffected. A given declination as determined by the trigon corresponds to a definite date (or, in general, two dates) and a modern construction would probably calibrate the trigon by the calendar rather than the conventional zodiacal signs, using data from an almanac.

In Figure 2 the noon hour-line is on the right, the summer part of the trigon is on the left and of the side declination scale IJ the positive half is above AB, and in use the top left corner is pointed towards the sun. Whether on the right or left, it is the midnight hour-line that is towards the sun. As will be shown later, the scale IJ can be transferred from Y to X without altering the summer and winter sides of the trigon scale, provided the IJ scale is inverted. Or, half can be on one side and half on the other, as in the diagram of the Navicula, Figure 7.

Trigonometrical Proof: Figure 3

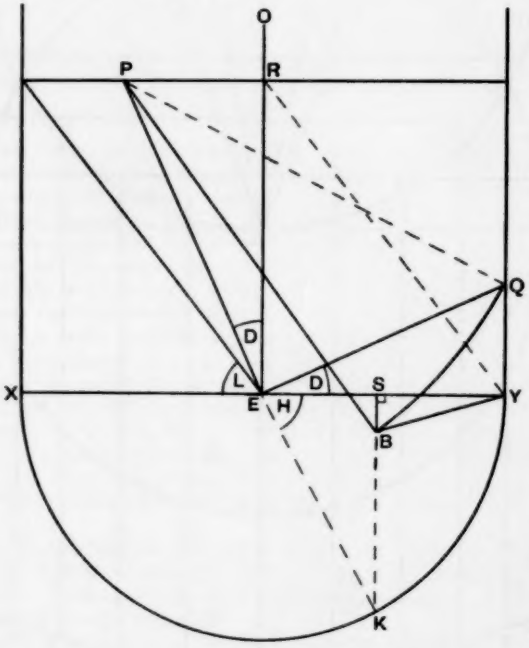


FIG. 3.

* "Early Science in Oxford" Vol. II. pp. 38–41. The Ms. Gunther described (Bodley 68) relates to the *Navicula de Venetiis*.

In Figure 3, P is the point of suspension of the string and Q the point on the side scale at which the bead is set. The construction has made $\widehat{PER}$ and $\widehat{QEY}$ equal to D and RE equal to $XE \tan L$. B is the position of the bead, so $PB = PQ$, the effective length of the string. Since PB is vertical when YX points towards the sun, $\widehat{PBS} = A$. When the hour angle is H, $ES = XE \cos H$. (H is $\widehat{YEK}$, where K is the intersection of SB produced with the semi-circle on XY.)

Now the triangles PRE and QYE are similar and so, therefore, are PQE and RYE.

Hence
$$\begin{aligned} PQ &= RY \sec D, \\ &= XE \sec L \sec D. \end{aligned}$$

The projection of PB on a line parallel to XY is

$$\begin{aligned} PB \sin A &= PQ \sin A \\ &= PR + ES \\ &= ER \tan D + XE \cos H, \\ &= XE \tan L \tan D + XE \cos H. \end{aligned}$$

Hence
$$\begin{aligned} \sin A &= \frac{XE}{PQ} \tan L \tan D + \frac{XE}{PQ} \cos H \\ &= \sin L \sin D + \cos L \cos D \cos H. \end{aligned}$$

Since this is identical with equation (1) it follows that the hour angle given by the dial is equal to the sun's hour angle when its altitude is A.

Geometrical Proof: Figure 4

The above proof, which set my mind at rest in 1950, soon lost its efficacy. It is hard to believe that Regiomontanus or his predecessors, lacking symbolic algebra and analytical trigonometry, would have used such an argument: but it is inconceivable that reasoning of this kind gave rise to the dial in the first place.

Many months later I hit on the following geometrical argument. In Figure 4, EzE' is the orthogonal projection of the celestial sphere on to a plane perpendicular to the equinoctial circle and through the centre O and the observer's zenith z. EE' is the equinoctial diameter and p, p' are the north and south celestial poles. Before projecting the rest of the sphere on to this plane, first project it gnomonically (in straight lines through O) on to the cylinder which touches the sphere along the equinoctial circle; then project the result on to the plane EzE'.

The observer's zenith and nadir project into Z and N.

The point where the meridian through z meets the sun's declination circle is d; its projection is D.

DD' is the projection of the declination circle. S is the projection of s, the sun's position on the sphere. We shall need to refer to S', the point on the circumscribing cylinder corresponding to s, before it is projected into S. S' should be imagined as vertically above S when the plane of the paper is horizontal, and s is on OS'.

SK and SM are perpendicular to OE and OZ respectively, and R is on SK produced such that $QR = QZ$.

Proof

The gnomonic projection from s to S' preserves the sun's hour angle, the angle between the plane of the paper and the plane $pp'S'$. Hence SKR is

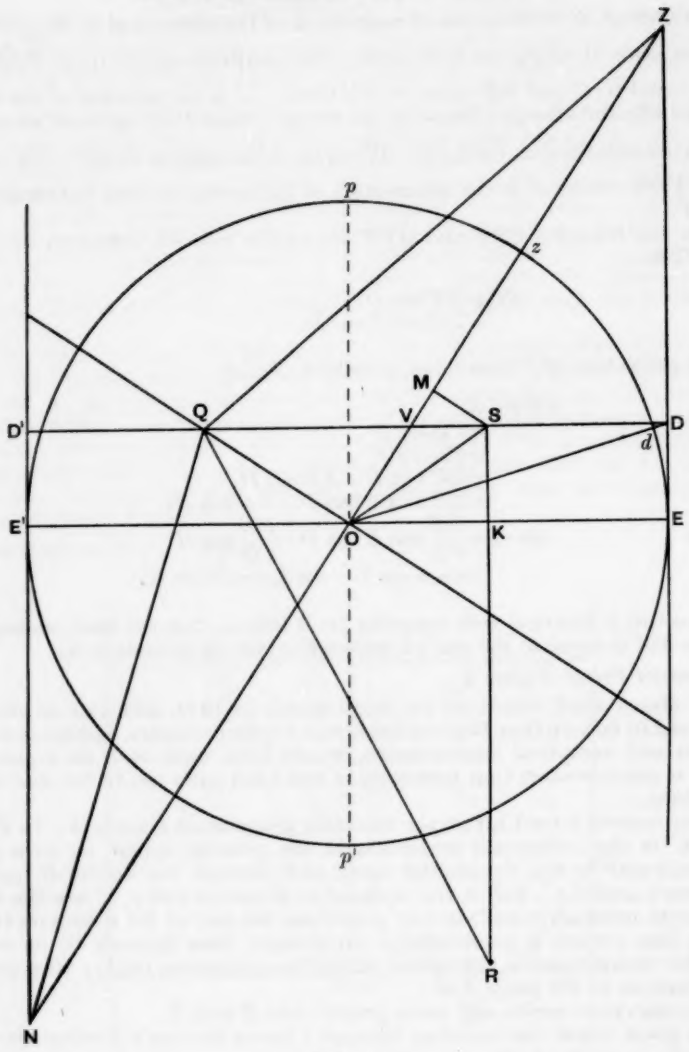


FIG. 4.

the hour-line for the sun, the scale of hours extending from E (noon) to E' (midnight).

Let OZ meet DD' in V.

Then, since SM is parallel to OQ, as S moves along DD' the ratio

$$\frac{OM}{QS}$$

remains constant and equal to

$$\frac{OV}{QV};$$

but since the points Q, Z, D, O are concyclic, the triangles QVZ and OVD are similar, so that this ratio is equal to

$$\frac{OD}{QZ}.$$

Hence
$$\frac{OM}{OD}=\frac{QS}{QZ}.$$

But $OD=OS'$, each being a generator of the cone whose base is the circle $DS'D'$ and vertex O ; while $QR=QZ$.

Hence
$$\frac{OM}{OS'}=\frac{QS}{QR}.$$

Therefore the triangles OMS' and QSR, both right-angled, are similar.

But $\widehat{MOS'}$ is the angle between the zenith and sun, the co-altitude.

Hence
$$\widehat{MS'O}=A=\widehat{QRS}.$$

Thus, when DD' is directed towards the sun a bead at R suspended on a string from Q is vertically below Q and lies on the appropriate hour-line.

The figure of this proof is that of the " reversed " dial, since Z is the point on the side *latitude* scale, $\widehat{EOD}$ is the declination and $\widehat{QOp}$ is the latitude.

All that, of course, is good fun, but bad mathematics. In theory the Greeks could have produced this argument ; but it is not very likely that they did. For the whole of the above derivation is *a posteriori*, a clear case of synthetic reasoning, and therefore no derivation at all. Had the diallers used this basis for their construction, the dial could legitimately be called an astrolabe, since it embodies a geometrical projection of the celestial sphere, with altitude and declination lines. It could have been ; but later analysis shows a more likely derivation.

The Reversed Dial

Before continuing the search for the dials' origin, it is necessary first to clear up some problems related to particular variants of the rectilinear dial.

A dial in the Deutsches Museum in Munich (No. 40575) is of the reversed type, with the trigon inverted too, forming an extremely compact dial.

It is immaterial whether the trigon in Figure 2 is above or below the line XY, provided the side declination scale is suitably adjusted. It is the position of the point of suspension in the XY direction that matters, not its height above or below XY. It is also immaterial whether half the side scale is " below left " or " above right ". In Figure 4, $QN=QZ$, so that the point N will serve equally as well as Z for setting the bead. (In this case it is the side *latitude* scale which moves from " above right " to " below left ".)

Now consider the effect of interchanging the positions of the declination and latitude scales. Declination is either positive or negative ; in Europe, latitude is positive. Hence only one half (left or right) of the trigon and one half of the side latitude scale need be drawn on a reversed dial. But whereas before the trigon was drawn for positive latitude only, it must now be drawn for positive and negative declinations, above and below XY. However, for negative declination $(-D)$ and latitude (L) can be substituted declination (D) and latitude $(-L)$, making the other half of the trigon do duty for negative declination.

The Munich dial is obtained from this double inversion, and is illustrated

in Figure 5. In the actual dial the noon hour-line is on the left and the sun on the right. I have reversed it to conform with the other drawings.

Local Dials

A rectilinear dial for constant latitude is simply constructed by eliminating those parts of a reversed dial which enable the latitude to be varied. The

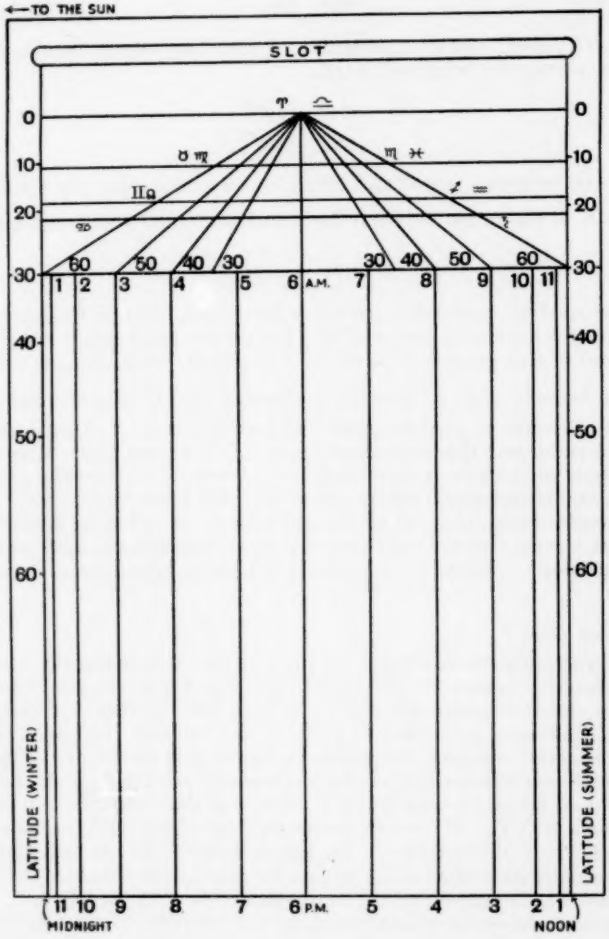


FIG. 5.

point of suspension of the string needs only to slide up and down the declination scale of the trigon along a line of constant latitude, the bead being set as usual at the fixed latitude point on the noon hour-line. Such a dial is in the Deutsches Museum (No. 60317) and is illustrated in Figure 6. My notebook shows only a rough sketch, and on reconstructing this dial accurately I found a discrepancy about which I had some doubts. From my sketch I

conjecture that the latitude must have been about 45°. If so, the 4 a.m. and 8 p.m. markings hardly apply. It may be that the very economical hour-line calibrations were extended beyond the summer sunset times to make for easier reading. Such an extension at the sunset-sunrise end of the scale turns extrapolation into interpolation, a more accurate process.

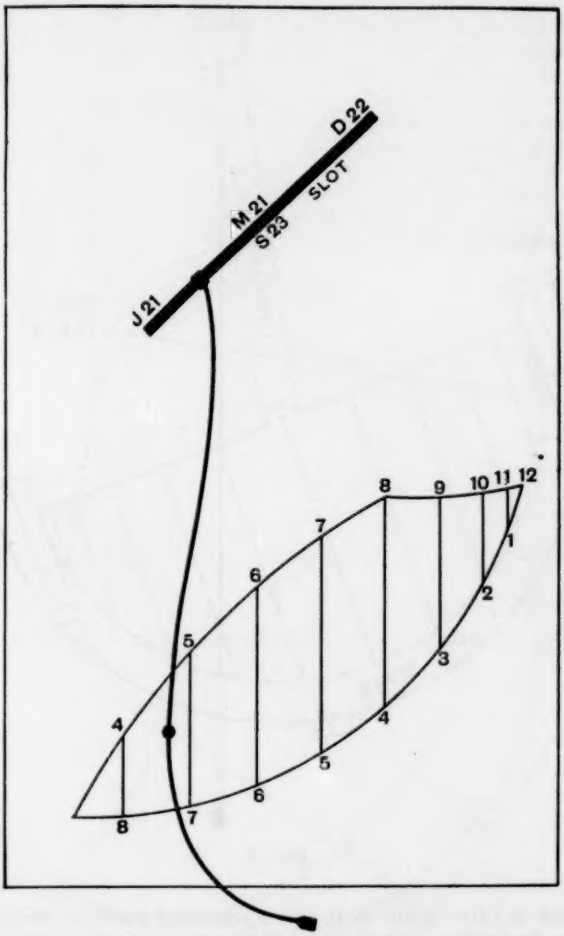


FIG. 6.

The Little Ship of Venice (*Navicula de Venetiis*)

This is an older form of the ordinary universal dial. Since some examples are dated from the 13th century it appears certain that Regiomontanus probably only recorded, perhaps modified, the construction, which may therefore have originated with the Greeks, being preserved by the Arabs and revived during the Renaissance.

Compare Figures 7 and 2. The hull of the ship is a semi-circle carrying the

hour-lines. The mast has a latitude scale and pivots about the centre of the semi-circle, the angle of inclination to the mid-position being the declination. A declination scale at the side is identical with that in Figure 2, but may be split in two parts, as shown.

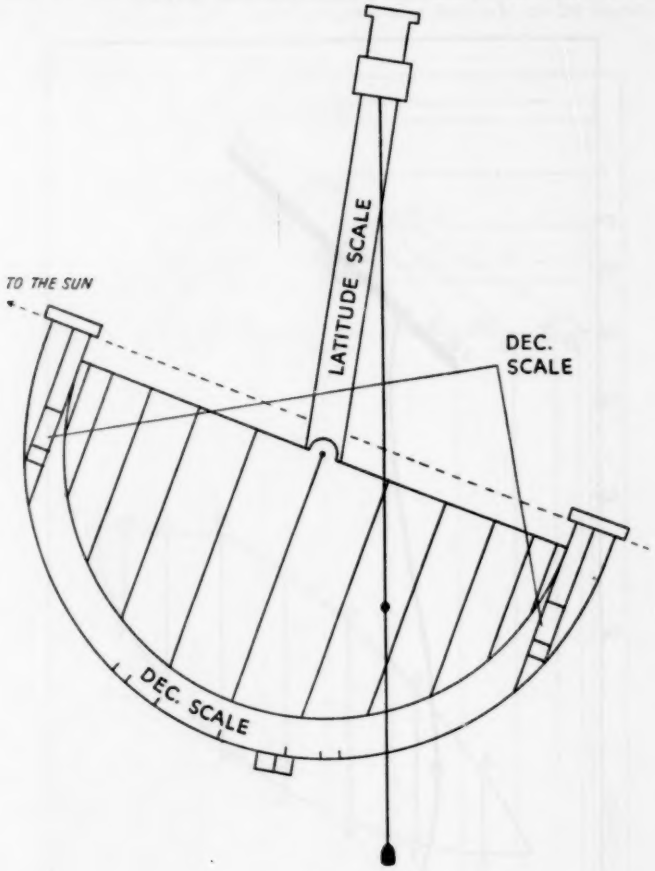


FIG. 7.

If the mast is tilted away from the equinoctial position through an angle equal to the declination, the distance of the point of suspension of the string along the mast must be increased in the ratio sec D : 1, otherwise even sunset time will not be correct. It would be possible to correct for this discrepancy by making the angle of tilt not the declination but

$$\sin^{-1}(\tan D);$$

but this would throw out the side scale by lowering the point of suspension. In the ship dial in the Whipple Museum dated 1620 and bearing the initials SF (Samuel Foster?) the correction is applied to the latitude scale on the mast. The equinoctial latitude scale from 0 to 60° is on the front of the mast; on

the back is a solstitial scale in the neighbourhood of 50° to 60° only. For intermediate dates no doubt the position was interpolated by estimation. (The difference between the equinoctial and solstitial settings at latitude 52° is about a quarter of an inch, the radius of the semi-circle being 1·9 inches.)

The Source of the Dials

The problem of determining by construction the sun's altitude for a given latitude, declination and time of day is practically identical with that of finding the great circle distance on the terrestrial globe between points of known latitude and longitude; and solutions to that problem were known to the Greeks. Ptolemy's analemma was constructed for this purpose. The following derivation seems to me to be of the kind which might very well have occurred to him or his contemporaries. The method is that of orthographic projection of the celestial sphere on to a diametral plane through the observer's zenith and the sun's meridian position. The first key to the solution has already been used in the geometrical proof in a modified form; it is the invariance under this projection of the height of the sun (on the model of the celestial sphere) above the horizon circle.

Determination of the Sun's Altitude: Figure 8

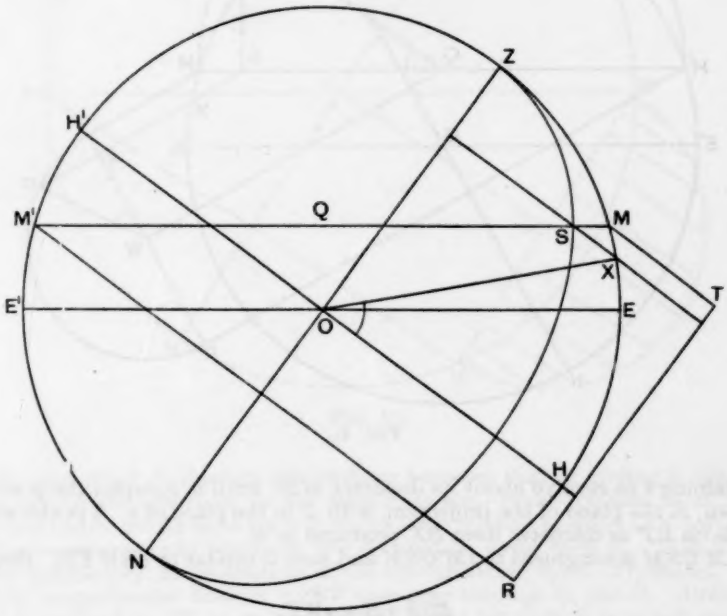


FIG. 8.

Let *O* be the centre of the sphere, *Z*, *N* zenith and nadir; *M* the meridian and *M'* the midnight position of the sun for a given declination. Then *MM'* is the projection of the sun's apparent diurnal path and *S* any point on this projection. *HOH'* is perpendicular to *ZON* and is the projection of the observer's horizon.

The height of the sun above this horizon is the distance from *S* to *OH*. If *s* is the sun's true position on the sphere, the ellipse on *ZN* as major axis

passing through S is the projection on to the diametral plane ZNH of the great circle through s.

If we draw SX parallel to OH to meet the circumference in X then the triangles ZXO and ZsO are congruent, so that the angle between XO and OH is the sun's altitude ($\widehat{ZOX} = \widehat{ZOs}$ is the co-altitude).

Determination of time of day

The position of S on MM' depends on $\widehat{MQs}$, the hour angle H. Further, since SX is parallel to OH, the position of SX relative to parallel lines through M and M' also determines the hour angle. This is the second key to the solution. Refer to Figure 9. Imagine the declination semi-circle

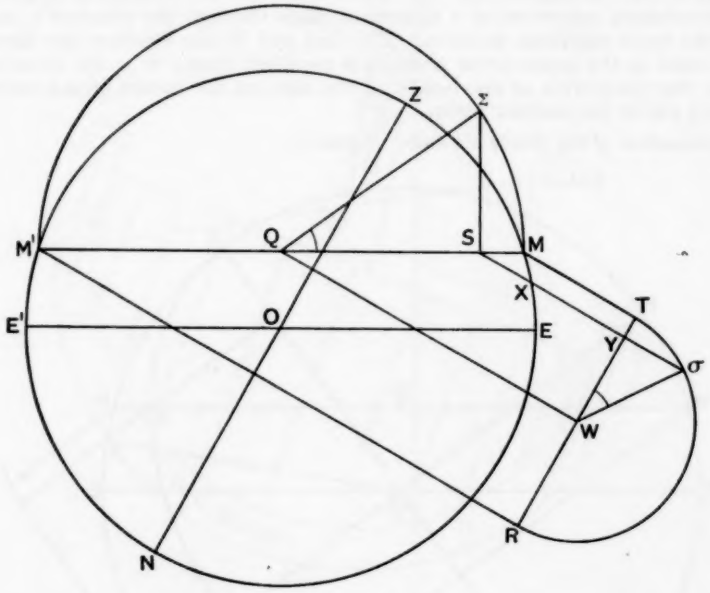


FIG. 9.

containing s to revolve about its diameter MM' until it occupies the position shown, in the plane of the projection, with Σ in the place of s. Let the semi-circle on RT as diameter meet SX produced in σ.

$\Sigma M'QSM$ is congruent to $sM'QSM$ and each is similar to $\sigma RWYT$. Hence,

$$\widehat{\Sigma QS} = H = \sigma\widehat{WT}.$$

The semi-circle on RT as diameter is divided into 12 equal parts to provide the hours, and the time can be read off from the position of SX with respect to this scale.

Development of the Construction for Repeated Use

To solve a series of problems by the above method would require a fresh figure for each latitude and declination. To simplify the procedure and avoid repeated labour the same projection of the hour-scale must always be used. That is, RT, the projection of MM' on a line parallel to ZN, must be invariant.

Hence the size of the main circle must vary while the shape of the figure remains the same.

Equinoctial Configuration

At the equinoxes $M'QM$ is a diameter and the figure reduces to that shown in Figure 10. Beginning with RM and the time-scale, make $\widehat{RMM'}$ equal to the latitude and $\widehat{MRM'}$ a right-angle. Then O, the mid-point of MM' is the centre of the base circle and the rest of the construction follows.

General Configuration

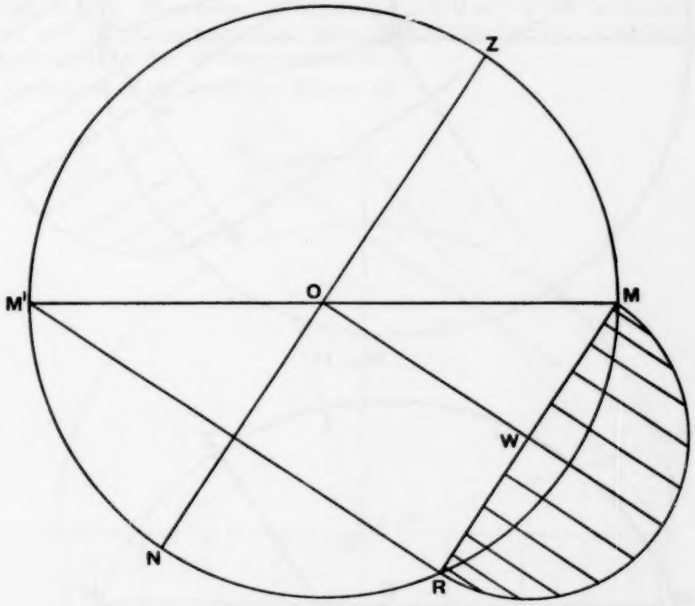


FIG. 10.

For any other declination the problem becomes that of finding a suitable position for the diameter of the time-scale semi-circle relative to O. An obvious choice would be to leave it in the same relative position as it occupied at the equinoxes, and construct the declination circle from it. This is not quite satisfactory, as an analysis of the corresponding figure (11) shows.

The perpendicular bisector of RT now goes through Q, not O. Although QM is inclined to RT at an angle equal to the latitude L it is not easy to reconstruct the figure beginning with RT because other angles involved are not exactly simple functions of D and L, although they are close approximations. For example, if W is the mid-point of RT, Q, O, W, E are concyclic and

$$\widehat{OWQ} = \widehat{OEQ},$$

which is nearly equal to the declination $(\widehat{MOE})$.

A slight modification makes this equality exact (Figure 12). Instead of

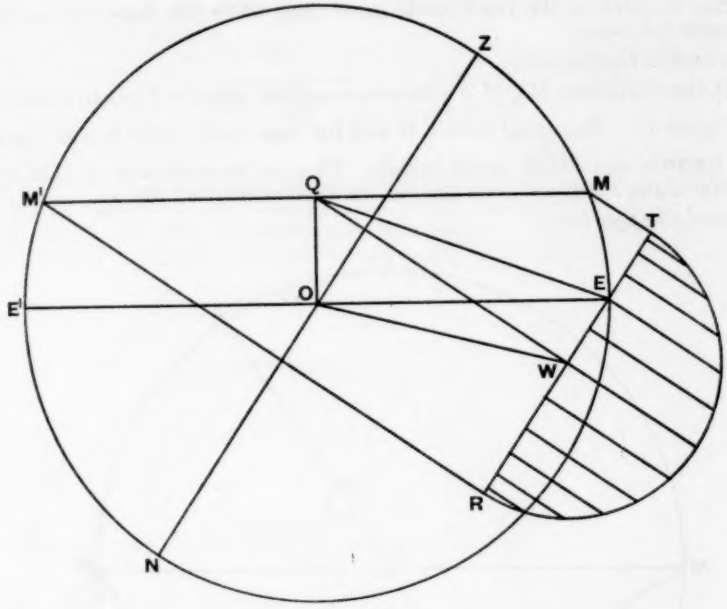


FIG. 11.

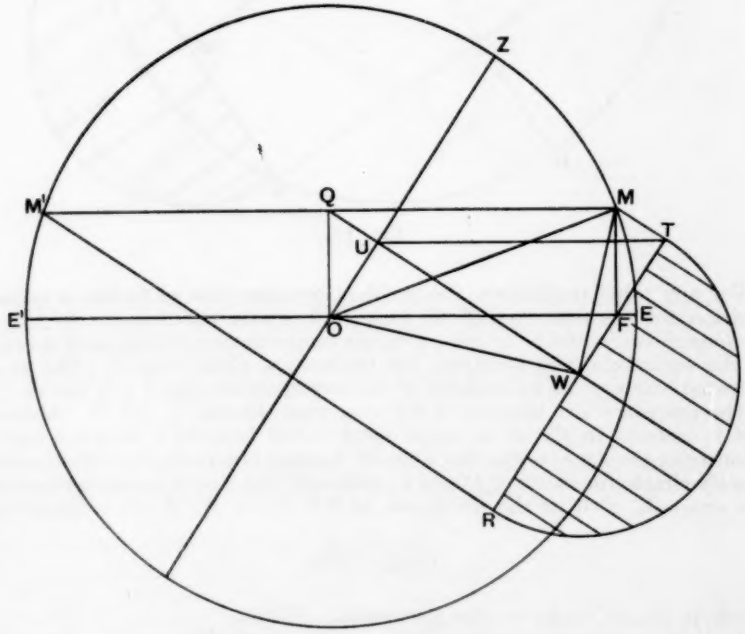


FIG. 12.

taking the projection RT through E, let the circle on OM as diameter meet the parallel through Q to $M'R$ and MT in W. Then Q, O, W, M are concyclic, and

$$O\widehat{W}Q = O\widehat{M}Q = D.$$

The fourth vertex F of the rectangle $OQMF$ lies on WT, since QF is a diameter of the circumscribing circle of the rectangle, W is on this circle, and $Q\widehat{W}T$ is a right-angle. Let QW meet OZ in U.

The triangles QOU and MFT are congruent, so that QU is equal and parallel to MT. Therefore UT is parallel to QM and $U\widehat{T}W$ is the latitude.

The rest of the construction then yields directly the essentials of the navicula and the dial of Regiomontanus.

The Derivation of the Navicula: Figure 13

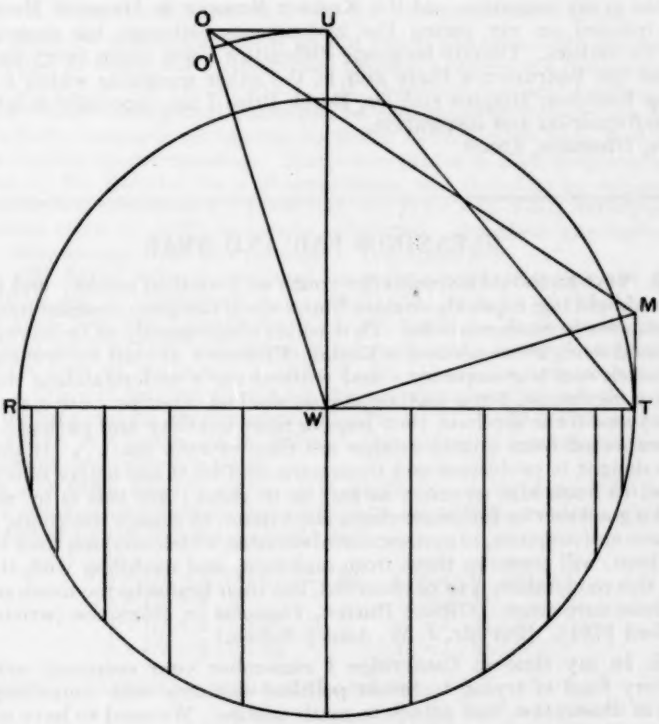


FIG. 13.

Draw the time-scale semi-circle on RWT as diameter.

Make $W\widehat{T}U$ equal to L and $T\widehat{W}U$ a right-angle.

Make $O\widehat{W}U$ and $M\widehat{W}T$ equal to D, and the angles at T and U right-angles.

Then O is the point of suspension, corresponding to the centre of the celestial sphere, and M the point on the side scale at which the bead is set.

A simple tilting of the mast brings U to O', not O. Hence, the equinoctial latitude scale along the mast needs to be stretched as the declination increases.

The Derivation of the Dial of Regiomontanus

The line UO in Figure 13 is the latitude line on the trigon, while $U\hat{W}O$ is the declination.

If this is in fact the line of development of this dial, it is not an astrolabe, since the lines in the construction are a projection not of the celestial sphere, but of different parts of it, according to latitude and declination. A more direct representation, with the top of the dial actually pointing to the sun and the plumb bob to the nadir, is obtained by projecting the celestial sphere on to a diametral plane through the observer's zenith and the sun's instantaneous position; but as this projection is less obvious and yields no important novelty it is left as an exercise to the reader.

I am indebted to the curators and their assistants at museums in England and Germany for the very ready and cheerful assistance which they have given me in my enquiries. At the Kestner Museum in Hanover Herr Küthmann insisted on my seeing the instruments although his museum was closed for repairs. Despite language difficulties I was given every facility to examine the instruments there and in the other museums which I visited. To Miss Kathleen Higgins and Dr. Derek Price I am especially indebted for much information and inspiration.

Kilsyke, Ulverston, Lancs. A.W.F.

GLEANINGS FAR AND NEAR

1872. Next he should acquaint the youth with mathematicks; and to invite to it, he should begin quickly to show him some of the more pleasant mechanical performances in mathematicks. That which is necessarily to be known to one that would study these sciences is Euclid's Elements, at least his first six books, arithmetick and trigonometry; and without one's understanding these, one may be a mechanist, but a mathematician shall he never be. For stereometry, algrebra and conic sections, they require more subtlety and patience, than is to be expected from youth, neither are they of such use.... If the youth have a delight in problemes and theoremes, and be of ane active fancy, it will be good to hook him as much as can be to them; for this is by wise men judged a good service for preserving a state quiet, to engage the young nobility who have active spirits, to mathematical sciences, which carrying their thoughts after them, will preserve them from ambition, and meddling with the state. But in this moderation is to be observed, lest their brains be too much stretched with these curiosities.—Gilbert Burnet, *Thoughts on Education* (written 1668, published 1761). [Per Mr. J. W. Ashley Smith.]

1873. In my time at Cambridge I remember your economic authorities were very fond of trying to invest political economy with something of the modes of illustration that attach to mathematics. We used to have economic laws described in curves and angles and other things naturally repugnant to a wholesome mind.—From a speech by W. S. Robson at Cambridge, quoted in Robson's life by G. W. Keeton, *A Liberal Attorney-General*. [Per Mr. P. C. Wickens.]

1874. I am full of projects what I shall do. The Fathers arise again full before me. This vacation I should not wonder if I took up the study of the Modern French Mathematics.—J. H. Newman to his mother, 1830; quoted by Geoffrey Faber, *The Oxford Apostles*, p. 237.

sphere, and M the point on the side scale at which the bead is set.

PYTHAGOREAN TRIADS IN BABYLONIAN MATHEMATICS

By E. M. Bruins

1. About ten years ago Neugebauer and Sachs published their Mathematical Cuneiform Texts in which the contents of Plimpton 322 were analysed for the first time. The numerical values contained in this text are given in Table I. The theory put forward by Neugebauer and Sachs was in my view not consistent with known facts and in the Proceedings of the Academy at Amsterdam I put forward a different interpretation. Since then I have made several short comments in connection with this text, but up to the present I have not given an explicit interpretation of the errors occurring on Plimpton 322. Recently R. J. Gillings has returned to this subject in his article entitled " The Oriental Influence on Greek Mathematics " published in the *Mathematical Gazette*, Vol. XXXIX, p. 187. I consider therefore that it may be of use to give here a short summary of the evidence relating to Pythagorean Triads in Babylonian Mathematics, and to add some hitherto unpublished remarks on the errors which Plimpton 322 contains, and my interpretation of them.

2. We begin by remarking that if we put one of the sides l of a right-angled triangle equal to unity, the Pythagorean relation between the remaining sides d and b is $d^2-b^2=(d+b)(d-b)=1$. If therefore we set $d+b=\lambda$, then $d-b=1/\lambda$. Now a reciprocal value can be calculated in Babylonian Mathematics *only* for numbers containing no prime factors other than 2, 3 and 5, i.e. for so-called *regular* numbers. Extensive tables of such reciprocals were calculated by the Babylonian mathematicians, and therefore by reference to such a table the numbers $d=\frac{1}{2}(\lambda+1/\lambda)$, $b=\frac{1}{2}(\lambda-1/\lambda)$, $l=1$, satisfying the Pythagorean relation, could be simply computed. Are there any indications that the Babylonians used this relation? Yes, there are.

(i) On BM 34568 several problems on right-angled triangles are solved. Most of these are the triangles 3, 4, 5 ; 20, 21, 29 ; 6, 8, 10 ; 8, 15, 17. These triangles arise from the formula for $\lambda=2$, $2\frac{1}{2}$, 3 and 4. $\lambda=3\frac{1}{2}$ is omitted because it does not lead to a finite sexagesimal reciprocal. Using the sexagesimal system from now on, we have e.g., if $\lambda=2\,.\,30$, $1/\lambda=0\,.\,24$, whence $d'=2\,.\,54$, $b'=2\,.\,6$, $1'=2$. If we now divide by the common factor 6 (which is equivalent to multiplying by 10), we obtain $d=29$, $b=21$, $l=20$. This points to a series of values of λ increasing by halves. For values of λ much greater than 4 the triangles become very acute.

(ii) On AO 6486, after a series of 11 problems on right-angled triangles and the Pythagorean theorem, there follows a set of four problems which can be represented by the following equations :

$\lambda+1/\lambda=2\,.\,0\,.\,0\,.\,33\,.\,20$	$(\lambda=1\,.\,0\,.\,45)$
$\lambda+1/\lambda=2\,.\,3$	$(\lambda=1\,.\,15)$
$\lambda+1/\lambda=2\,.\,5\,.\,26\,.\,40$	$(\lambda=1\,.\,21)$
$\lambda+1/\lambda=2\,.\,0\,.\,15$	$(\lambda=1\,.\,4)$.

The solution is obtained by means of the relation

$$\left(\frac{\lambda+1/\lambda}{2}\right)^2=1+\left(\frac{\lambda-1/\lambda}{2}\right)^2$$

An interpretation of these values of λ can be easily given if we ask for triads with $1<\lambda<2$ of the form $\lambda=1+1/n$. Then both n and $n+1$ must be regular numbers, and the set of such numbers n consists of 1, 2, 3, 4, 5, 8, 9, 15, 24, 1 . 20. The corresponding values of λ are 2 ; 1 . 30 ; 1 . 20 ; 1 . 15 ; 1 . 12 ; 1 . 7 . 30 ; 1 . 4 ; 1 . 2 . 30 ; 1 . 0 . 45 ; which series explains the curious values of λ occurring on AO 6484.

3. Now let us apply these attested relations to the case of Plimpton 322. We first of all notice that the numbers for hypotenuse and side, apart from the much-used triangle 1 . 15, 45, are all relatively prime with respect to the factors 2, 3 and 5; in fact, accidentally, they are all completely co-prime. Further, if we calculate backwards to find the values of λ which would give rise to the numbers on the table, we are struck by the fact that in the range from $\lambda = 2 \,.\, 24$ to $\lambda = 1 \,.\, 48$ "all" regular numbers are present which have at most four sexagesimal places and whose reciprocals also have at most four sexagesimal places. These values are given in Table II. It now becomes clear why the numbers were indicated with ki 1, ki 2, ... ki 15 (i.e. the first, the second, ... the fifteenth). These numbers can only indicate the order in which the values of λ appear in a Table of Reciprocals having at most four places in one of the columns, and selecting only those numbers which have not more than four places in *either* column.

Since in this way an interpretation is possible which is completely in accordance with Babylonian methods used in other texts, we see no reason for the assumption that, in order to compute the numbers in this text, very irregular values of p and q were chosen in order that, after computing $d = p^2 + q^2$ and $b = p^2 - q^2$, an approximately linear increase of the ratio d/b would be obtained. In our opinion just the theory rejected by Neugebauer and Sachs—because they failed to find a method of deducing *their* values of p and q from the reciprocals—shows exactly the procedure expected to be used by the Babylonians. In my opinion also the explanation of the errors given by Gillings, starting from the values of p and q given by Neugebauer and Sachs, cannot be maintained. He follows Neugebauer and Sachs in assuming that the value 7 . 12 . 1 is an error caused by "putting the square of 2 . 41 instead of the number itself", and in order to explain the number 3 . 12 . 1 in place of the correct number 1 . 20 . 25 he has to assume that the scribe made an error in addition *and* an error in squaring. If one considers the skill which the scribe shows in the computation of 15 squares in the first column of the same tablet, and if further one takes into account that the columns in which the errors occur should not contain squares at all, we think that one has to look for other interpretations.

4. The Errors in Plimpton 322.

(*a*) The list of errors given by Neugebauer and Sachs is not a complete one. In addition to this they gave in their Mathematical Cuneiform Texts a series of units enclosed between brackets. Later on, after I had put forward my theory, some of the units were given without brackets, as if they appeared on the tablet. I went specially to New York to take Plimpton 322 in my hands and found that the first "unit" appearing on the later photographs is the "head" of the horizontal line between the rows, and that it is not a vertical wedge. In fact no units are present.

(*b*) In the first column of Plimpton 322 the system of transcription used by Neugebauer and Sachs, which indicates also the intermediate zeros, contains two errors: line 1; 59 . 15 instead of 59 . 0 . 15
line 13: 27 . 3 . 45 instead of 27 . 0 . 3 . 45;
whereas in line 8 the number 41 . 33 . 59 . 3 . 45 is incorrectly given by the scribe and it should read 41 . 33 . 45 . 14 . 3 . 45.

The work involved in computing the squares contained in the first column —which gives $t^2 = \left(\frac{\lambda - 1/\lambda}{2}\right)^2$—is much less than one would expect. Considerably more effort would be required to compute $\left(\frac{\lambda + 1/\lambda}{2}\right)^2$, which is one unit greater than t^2. The set of t's in fact consists of one one-place value, six

two- and six three-place values, and two four-place values. The six two-place values can be easily verified :

line 1 : $59 . 30^2 = (1 - 0 . 0 . 30)^2 = 1 - 0 . 1 + 0 . 0 . 15 = 59 . 0 . 15$ (!)
line 5 : $54 . 10^2 = (54 + 0 . 10)^2 = 48 . 36 + 0 . 18 + 0 . 0 . 1 . 40$
$= 48 . 54 . 1 . 40$
line 6 : $53 . 10^2 = (53 + 0 . 10)^2 = 46 . 49 + 0 . 17 . 40 + 0 . 0 . 1 . 40$
$= 47 . 6 . 41 . 40$
line 9 : $48 . 6^2 = (48 + 0 . 6)^2 = 38 . 24 + 0 . 9 . 36 + 0 . 0 . 0 . 36$
$= 38 . 33 . 36 . 36$
line 13 : $40 . 15^2 = (40 + 0 . 15)^2 = 26 . 40 + 0 . 20 + 0 . 0 . 3 . 45$
$= 27 . 0 . 3 . 45$(!)
line 15 : $37 . 20^2 = (37 + 0 . 20)^2 = 22 . 49 + 0 . 24 . 40 + 0 . 0 . 6 . 40$
$= 23 . 13 . 46 . 40$

Let us now consider the case of line 8.

$$49 . 56 . 15^2 = (50 - 0 . 3 . 45)^2 = 41 . 40 + 0 . 0 . 0 . 14 . 3 . 45 - 0 . 6 . 15$$
$$= 41 . 40 . 0 . 14 . 3 . 45 - 0 . 6 . 15 = 41 . 33 . 45 + 0 . 0 . 0 . 14 . 3 . 45$$
$$= 41 . 33 . 45 . 14 . 3 . 45.$$

By adding at the wrong sexagesimal the scribe obtained 41 . 33 . 59 . 3 . 45 which contains one sexagesimal place too few. In this way the error is simply explained. If we now use the rules : " To multiply by 30 divide by 2 and shift the sexagesimal ", and " To multiply by 20 divide by 3 and shift the sexagesimal ", we can calculate the squares of the remaining three-place values quite simply, using the decompositions

line 3 : $57 . 30 . 45 = 57 . 30 + 0 . 0 . 45$
line 4 : $56 . 29 . 4 = 56 . 30 - 0 . 0 . 56$
line 7 : $50 . 54 . 40 = 51 - 0 . 5 . 20$
line 12 : $41 . 58 . 30 = 42 - 0 . 1 . 30$
line 14 : $39 . 21 . 20 = 40 - 0 . 38 . 40 = 40 - (0 . 40 - 0 . 1 . 20)$.

We think that after having this evidence of the scribe's skill in computing the squares in the first column of the table we can hardly maintain that he would make a mistake in squaring 1 . 31!

(*c*) In discussing the error in the last line, we observe that from $\lambda = 1 . 48$ we obtain in succession $1/\lambda = 33 . 20$, $\lambda + 1/\lambda = 2 . 21 . 20$, $\lambda - 1/\lambda = 1 . 14 . 40$. Since these have the common factor 20 we divide it out—i.e. we multiply by 3 and obtain in turn :

	7 . 4	3 . 44
dividing by 2	3 . 32	1 . 52
" " "	1 . 46	56
" " "	53	28.

We think we are not very likely to be wrong in assuming that the scribe factorised the numbers separately and so ran the risk of combining " the wrong lines " : in this case 53 and 56 instead of 53 and 28. This would easily occur if for example he divided 7 . 4 immediately by 4, but 3 . 44 in two steps of 2.

(*d*) The computation for line 2 would run as follows :

$\lambda = 2 . 22 . 13 . 20$, $1/\lambda = 25 . 18 . 45$;

$\lambda + 1/\lambda = 2 . 47 . 32 . 5$		$\lambda - 1/\lambda = 1 . 56 . 54 . 35$
divide by 5	33 . 30 . 25	23 . 22 . 55
" " "	6 . 42 . 5	4 . 40 . 35
" " "	1 . 20 . 25	56 . 7
" " "	16 . 5	
" " "	3 . 12 + 1	

The number 3 . 12 . 1 should be read as 3 . 13 (!) and is again the result of " taking the wrong line ".

(*e*) Finally we come to the error in line 13. Here we have $\lambda = 1 . 52 . 30$, $1/\lambda = 0 . 32$, $\lambda + 1/\lambda = 2 . 24 . 30$, $\lambda - 1/\lambda = 1 . 20 . 30$. Dividing by 30—i.e. multiplying by 2—we obtain immediately 4 . 49 and 2 . 41. Here again the same kind of mistake has been made. The factor for division would be read from the last sexagesimal and often a scribe makes little difference between 20 and 30. If such a mistake occurred in writing or reading, and the number 2 . 24 . 20 were taken instead of 2 . 24 . 30, a multiplication by 3 would give 7 . 12 + 1.

We think that in this way the errors in Plimpton 322 can be given a natural explanation ; without any " acrobatics " they are shown to be caused by the simple mistakes of " looking at the wrong line " and misreading the numbers 20 and 30.

We also conclude that the assumption that the scribe pushed the simplification of the numbers as far as possible by dividing out all regular factors—apart from the trivial factors of 1 . 15 and 45—is ruled out by the two cases of 56 in line 15 (even if we correct it to 28!) and 1 . 5 in line 5.

E.M.B.

TABLE I

[59 . 0 .] 15	1 . 59	2 . 49	ki 1
[56 . 56 .] 58 . 14 . 50 . 6 . 15	56 . 7	3 . 12 . 1	ki 2
[55 . 7 .] 41 . 15 . 33 . 45	1 . 16 . 41	1 . 50 . 49	ki 3
53 . 10 . 29 . 32 . 52 . 16	3 . 31 . 49	5 . 9 . 1	ki 4
48 . 54 . 1 . 40	1 . 5	1 . 37	ki[5]
47 . 6 . 41 . 40	5 . 19	8 . 1	[ki 6]
43 . 11 . 56 . 28 . 26 . 40	38 . 11	59 . 1	ki 7
41 . 33 . 59 . 3 . 45	13 . 19	20 . 49	ki 8
38 . 33 . 36 . 36	9 . 1	12 . 49	ki 9
35 . 10 . 2 . 28 . 27 . 24 . 26 . 40	1 . 22 . 41	2 . 16 . 1	ki 10
33 . 45	45	1 . 15	ki 11
29 . 21 . 54 . 2 . 15	27 . 59	48 . 49	ki 12
27 . 0 . 3 . 45	7 . 12 . 1	4 . 49	ki 13
25 . 48 . 51 . 35 . 6 . 40	29 . 31	53 . 49	ki 14
23 . 13 . 46 . 4[0]	56	53	ki[15]

The column of units between brackets given by Neugebauer and Sachs in the first column have been omitted for reasons stated in § 4*a*.

TABLE II

1	2 . 24	25	6	2 . 13 . 30	27	11	2	30
2	2 . 22 . 13 . 20	25 . 18 . 45	7	2 . 9 . 36	27 . 46 . 40	12	1 . 55 . 12	31 . 15
3	2 . 20 . 37 . 30	25 . 36	8	2 . 8	28 . 7 . 30	13	1 . 52 . 30	32
4	2 . 18 . 53 . 20	25 . 55 . 12	9	2 . 5	28 . 48	14	1 . 51 . 6 . 40	32 . 24
5	2 . 15	26 . 40	10	2 . 1 . 30	29 . 37 . 46 . 40	15	1 . 48	33 . 20

THE DECISION PROBLEM*

By R. L. Goodstein

There are many questions in mathematics to which an answer can be found by a purely mechanical procedure. For instance we can determine the valid statements of the class of the statements

n is prime

by the sieve of Eratosthenes ; the numbers 2, 3, 4 and so on up to any assigned n are written in a row, and we begin by striking out every second number, starting with 2. Of those which are left, the first is 3 and 3 is prime. Starting with 3, we then strike out every third number, and the first of those which remain is 5, which is prime. Starting with 5 we strike out every fifth number, determining the prime 7, and so on.

Such a mechanical procedure for deciding the validity of a class of sentences is called a *decision procedure* for the class. The *decision problem* for a class of sentences is the problem of finding whether or not a decision procedure for the class exists, and exhibiting the procedure when there is one. If a decision procedure for a class is known, the class is said to be *decidable* ; if on the other hand it can be shown that no decision procedure is possible, then the class is said to be *undecidable*.

Of course the decision problem is of interest only if the class of sentences concerned is sufficiently rich ; for instance when the class comprises all the statements of some branch of mathematics, or ideally the whole of mathematics.

The branches of mathematics for which the decision problem has been solved are characterised, not only by their mathematical but also by their logical content. An account of the decision problem must therefore be prefaced by some remarks, necessarily brief, about the concepts and notation of symbolic logic.

The basis of symbolic logic is sentence logic which studies the *combinations* of sentences which are valid independently of the validity of the constituent sentences. The elements of sentence logic are sentence variables $p, q, r, \ldots$; negation "$\neg$", and the logical connectives "and", "or" and "implies" denoted by "$\&$", "$\vee$" and "$\rightarrow$" respectively. A sentence is either a sentence variable or has one of the forms $\neg S$, $S \,\&\, T$, $S \vee T$, $S \rightarrow T$ where S, T are sentences. Sentence logic may be formulated either as an axiomatic theory or by means of truth tables.

If we take "$p \rightarrow q$" as an abbreviation for "$\neg p \vee q$" and "$p \,\&\, q$" as an abbreviation for $\neg(\neg p \vee \neg q)$, sentence logic may be based on the 4 axioms

$$(p \vee p) \rightarrow p, \quad p \rightarrow (p \vee q), \quad (p \vee q) \rightarrow (q \vee p), \quad (p \rightarrow q) \rightarrow (r \vee p \rightarrow r \vee q)$$

and the inference rule

$$\frac{S \qquad S \rightarrow T}{T}$$

permitting the derivation of T from proved sentences S and $S \rightarrow T$.

A proof in sentence logic is a finite sequence of sentences each of which is either an axiom, or is derived from an axiom by substituting sentences for sentence variables, or is derived from previous sentences of the proof by the rule of inference.

* A lecture delivered at the 1955 meeting of the British Association for the Advancement of Science.

Calculation with & and $\vee$ may readily be shown to follow the associative, commutative and distributive laws (like + and . in arithmetic). Writing p for $\neg\neg p$, $\neg p \vee \neg q$ for $\neg(p \,\&\, q)$, $\neg p \,\&\, \neg q$ for $\neg(p \vee q)$ and $\neg p \vee q$ for $p \to q$ we may transform any sentence into an equivalent sentence in conjunctive normal form

$$\phi_1 \,\&\, \phi_2 \,\&\, \ldots \,\&\, \phi_n$$

where each ϕ_i is a disjunction of variables or negated variables. By transforming $\neg S$ into conjunctive normal form S may be brought into the disjunctive normal form

$$\psi_1 \vee \psi_2 \vee \ldots \vee \psi_n$$

where each ψ_i is a conjunction of variables or negated variables.

Instead of postulating axioms we may define the logical operations by so-called truth tables.

The truth tables for negation and the connectives are as follows, T and F standing for true and false respectively.

$\neg$	p	&	$\vee$	$\to$	q
F	T	T	T	T	T
F	T	F	T	F	F
T	F	F	T	T	T
T	F	F	F	T	F

To find for example the value of $p \,\&\, q$ for assigned values of p and q we look in the column headed & and in the row in which the assigned values of p and q occur. In the third row for instance we see that $p \,\&\, q$ has the value false if p is false and q is true.

It may be shown that all provable sentences are universally valid, i.e. take the value T for all values of their variables, and conversely that every universally valid sentence is provable, so that the axiom system is *complete* with respect to the truth tables.

Sentence logic is itself an excellent example of a decidable system, the truth tables providing a decision procedure. For instance to test whether the sentence

$$\neg(p \to q) \to (p \,\&\, \neg q)$$

is universally valid or not we calculate the value T or F of the sentence for each assignment of values T, F for the variables p, q, the calculation being set out as follows:

$\neg$	$(p$	$\to$	$q)$	$\to$	$(p$	&	$\neg$	$q)$
F	T	T	T	T	T	F	F	T
T	T	F	F	T	T	T	T	F
F	F	T	T	T	F	F	F	T
F	F	T	F	T	F	F	T	F

The value T is obtained (in column five) for all values of p and q so that the sentence is universally valid.

It is quite easy to give an even simpler decision procedure for sentence logic. If we take the values of the sentence variables to be 0, 1 (the former standing for truth and the latter falsity) and if we write $1-p$, $p+q$, $p \cdot q$, $(1-p)q$ for $\rightarrow p$, $p \,\&\, q$, $p \vee q$ and $p \rightarrow q$ respectively, then to test the validity of a sentence we have only to work out the value of its representation to find if it is zero or not. The sentence considered above is represented by the formula

$$[1-\{1-(1-p)q\}]\{p+(1-q)\}.$$

If this takes the value unity, each factor must be a unit and so

$$\begin{aligned} p+(1-q) &= 1 \\ (1-p)q &= 1\,; \end{aligned}$$

from the second we see that $q=1$, $p=0$ and this contradicts the first, showing again that the sentence is universally valid.

The conjunctive normal form provides yet another decision procedure. The given sentence

$$\rightarrow(p \rightarrow q) \rightarrow (p \,\&\, \rightarrow q)$$

may be transformed into

$$\rightarrow\rightarrow(\rightarrow p \vee q) \vee (p \,\&\, \rightarrow q)$$

and thence into $\quad (\rightarrow p \vee q) \vee (p \,\&\, \rightarrow q)$

and finally into $\quad (\rightarrow p \vee q \vee p) \,\&\, (\rightarrow p \vee q \vee \rightarrow q),$

and since each bracket contains a variable and its negation the sentence is recognised as universally valid.

In sentence logic we operate upon sentences as unanalysed wholes. In the next stage in the development of logic we proceed to analyse sentences into subject and predicate. In predicate logic, as this second stage is called, we distinguish three classes of variables

(i) Sentence variables $p, q, r, \ldots$
(ii) Individual variables $x, y, z, \ldots$
and (iii) Predicate variables $P(x)$, $Q(x, y)$, $R(x, y, z)$, ...

Predicates may be valid for all values of their arguments or for some, or for none. If a predicate $P(x)$ is valid for all values of its argument we write

$$Ax\,P(x)$$

(read " for all x, $P(x)$ "), and if a predicate $P(x)$ is valid for some values of its argument we write

$$Ex\,P(x)$$

(read " there is an x such that $P(x)$ ").

The operators A, E are called quantifiers; A is the universal and E the particular or existential quantifier. A sentence of predicate logic is either a sentence variable, or a predicate variable, or a combination of sentences $\rightarrow S$, $S \,\&\, T$, $S \vee T$, $S \rightarrow T$ or takes one of the forms $Ax\,P(x)$, $Ex\,Q(x)$.

As axioms for predicate logic we may take

$$Ax\,P(x) \rightarrow P(y), \quad P(y) \rightarrow Ex\,P(x)$$

and for drawing inferences the rules

$$\frac{S \to P(x)}{S \to Ax\, P(x)}, \qquad \frac{P(x) \to S}{Ex\, P(x) \to S}$$

(where x is not a free variable in S) reaffirming the rule

$$\frac{\begin{matrix} S \\ S \to T \end{matrix}}{T},$$

the sentence below the line being inferred from that (those) above by the rule in question.

Taking for granted some obvious rules about changing variables we may say that a predicate sentence is provable if it is an axiom, or a valid sentence of sentence logic, or is derived from a provable sentence by substituting sentences for sentence variables or is derived from provable sentences by the rules of inference. An important property of both predicate logic and sentence logic is the *deduction theorem* which says that if a sentence T is derivable from a hypothesis S (no substitution in the variables in S occurring in the derivation) then $S \to T$ is a provable sentence.

As an example of the use of the deduction theorem we prove the sentence

$$(Ax)\{P(x) \to Q(x)\} \to \{(Ex)\, P(x) \to (Ex)\, Q(x)\}.$$

From the hypothesis	$(Ax)\{P(x) \to Q(x)\}$
and the axiom	$(Ax)\{P(x) \to Q(x)\} \to \{P(y) \to Q(y)\}$
we derive	$P(y) \to Q(y)$
and hence, by the axiom	$Q(y) \to (Ex)\, Q(x)$
we derive	$P(y) \to (Ex)\, Q(x)$
from which we infer	$(Ex)\, P(x) \to (Ex)\, Q(x)$

and by the deduction theorem conclude

$$(Ax)\{P(x) \to Q(x)\} \to \{(Ex)\, P(x) \to (Ex)\, Q(x)\},$$

as desired.

To provide a content for the sentences of predicate logic we suppose given some domain of individuals to which the individual variables and the quantifiers A and E refer. A value of a predicate variable is simply an assignment of a truth value for each assignment of individuals from the domain for the individual variables in the predicate. For instance in a domain of two individuals 0, 1 there are just 4 predicates with one argument, 16 with two arguments, 256 with three arguments ; each predicate with two arguments is given by a table like

	0	1
0	T	F
1	F	T

The predicate given by this table, $P(x, y)$, say, is such that $P(0, 0)$ is true, $P(0, 1)$ is false, etc. The sixteen ways of filling in the T's and F's in the table give the sixteen possible predicates with two arguments.

A sentence in predicate logic is said to be *valid* in a given domain if it is true for every assignment of predicates and individuals from the given domain, and to be *satisfiable* in a given domain if it is true for some selection

of predicates and individuals. A sentence valid in every domain is said to be universally valid, and a sentence satisfiable in some domain is said to be satisfiable. Thus either a sentence S is universally valid or its negation $\rightarrow S$ is satisfiable. All provable sentences are universally valid and the converse is also true, so that predicate logic is also complete with respect to truth values.

Predicate logic is an example of an undecidable system, but various subsystems of predicate logic are decidable. In particular monadic predicate logic, the logic of predicates with a single argument, is easily shown to be decidable.

Any predicate sentence (monadic or not) is decidable in a specified finite domain since in such a domain the concepts of " all " and " some " are equivalent simply to finite conjunctions and disjunctions. The special feature of monadic predicates is that a sentence containing k monadic predicate variables is universally valid if and only if it is valid in a domain with 2^k elements.

Let us suppose that some sentence S containing k monadic predicate variables is satisfiable in a domain D, containing more than 2^k individuals, by some choice $p_1, p_2, \ldots, p_k$ of predicates for the predicate variables $P_1, P_2, \ldots, P_k$ in S. We separate the elements of D into classes, two elements going into the same class only if the values of the predicates $p_1, p_2, \ldots, p_k$ are the same for the two elements. Since there at most 2^k different arrangements of truth values for k predicates, the elements of D fall into at most 2^k classes, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ say, $n \leqslant 2^k$. Let e_r be an element of D in the class α_r, for $r = 1, 2, \ldots, n$ and for these values of r we define the predicate $q_i(r)$ to have the same truth value as $p_i(e_r)$, $i = 1, 2, \ldots, k$, in the domain F with elements $1, 2, 3, \ldots, n$; then by the definition of the q_i, sentence S is satisfiable in F by the predicates q_i if and only if S is satisfiable in D by p_i. The domain F may contain less than 2^k individuals but if S is satisfiable in F it will also be satisfiable in a larger domain with exactly 2^k individuals. By applying the foregoing discussion to $\rightarrow S$, we conclude that S is universally valid, if and only if it is valid in a domain of 2^k individuals.

Any predicate sentence may be expressed in so-called prenex normal form in which all quantifiers occur un-negated at the beginning and all operate on the same combination of predicates; this may be achieved by replacing each instance of a negated quantifier $\rightarrow Ax\, P(x)$ and $\rightarrow Ex\, P(x)$ by the equivalent forms $Ex \rightarrow P(x)$ and $(Ax) \rightarrow P(x)$ respectively, and then relettering the variables so that each quantifier has its proper variable and may be brought to the front of the sentence. The set of quantifiers in a prenex form is called the *prefix*, and the combination of predicates operated upon the *matrix* of the form.

Decision procedures are known for the universal validity of classes of sentences with a number of special prefixes. These comprise prefixes consisting solely of universal quantifiers or solely of particular quantifiers; prefixes in which all the universal precede all the existential quantifiers and prefixes with just one particular quantifier or two particular quantifiers not separated by a universal quantifier. For some other prefixes decision procedures are known only for special types of matrices, and for certain prefixes like $(Ex)(Ay)(Ez)$ and $(Ex)(Ay)(Ez)(Eu)$ the problem remains open.

I want to turn now from these purely logical instances of decidable systems to consider the decision problem for algebra. A particular fragment of algebra, called elementary algebra, consisting of the theory of polynomials of specified degree in any assigned number of real variables with integral coefficients is known to be decidable. The matrix of any sentence of elementary algebra in prenex form is a combination of equalities and inequalities between polynomials which may readily be brought into a disjunction of sentences of the form

$$\alpha_1 = 0 \;\&\; \alpha_2 = 0 \;\&\; \ldots \;\&\; \alpha_m = 0 \;\&\; \beta_1 > 0 \;\&\; \beta_2 > 0 \;\&\; \ldots \;\&\; \beta_n > 0$$

where each α and β is a polynomial of specified degree.

The decision procedure for elementary algebra is a method for eliminating quantifiers, transforming a sentence with quantifiers into an equivalent sentence without quantifiers or variables, the truth or falsehood of which may of course be determined simply by trivial calculations and sentence logic.

In essence the elimination procedure is a uniform method for determining the condition on the coefficients necessary and sufficient for the existence of a root of a set of polynomial equalities and inequalities. For instance the sentence

$$(Ay)(Ex)\{a_0+a_1x+a_2x^2+(b_0+b_1y+b_2y^2)x^3>0\}$$

(where the a's and b's are integers) is equivalent to

$$(Ay)\{(a_0>0)\vee(a_1^2>4a_0a_2)\vee(a_2>0)\vee\neg(b_0+b_1y+b_2y^2=0)\}$$

which in turn is equivalent to

$$(a_0>0)\vee(a_1^2>4a_0a_2)\vee(a_2>0)\vee(4b_0b_2>b_1^2)\vee\}\neg(b_0=0)\ \&\ (b_1=0)\ \&\ (b_2=0)\}.$$

Elementary algebra is concerned solely with the fundamental field operations. Algebraic concepts whose definition involves set theoretic notions, or the notion of a natural number are excluded. Thus for instance although the sentence " every cubic has a root " belongs to elementary algebra, the sentence " every polynomial has a root " does not.

The exclusion of natural number variables is essential to the proof of decidability for, as we are now going to show, any system which contains natural number variables and both the operations of addition and multiplication is undecidable.

We start by considering a fragment of arithmetic, which we shall call system $\mathscr{A}$, based on predicate logic and the axioms

1. $Sx=Sy\rightarrow x=y$
2. $\neg(0=Sy)$
3. $\neg(x=0)\rightarrow Ey(x=Sy)$
4. $x+0=x,\quad x+Sy=S(x+y)$
5. $x\,.\,0=0,\quad x\,.\,Sy=x\,.\,y+x.$

The variables in these axioms are natural number variables, Sx symbolising the next number after x, so that the natural numbers themselves are represented in the system by 0, $S0$, $SS0$, $SSS0$, and so on. We shall abbreviate the sign $SSS\ldots S0$ with p S's to N_p. The first axiom says that different numbers have different successors, and the second and third say that every number except 0 is the successor of some number. The groups of axioms 4 and 5 define addition and multiplication.

System $\mathscr{A}$ contains a certain amount of arithmetic but there are many simple theorems which are not provable in it. For instance none of

$$\text{(i)}\quad x\neq Sx,\quad 0+x=x,\quad x+y=y+x,\quad 0\,.\,x=0,\quad x\,.\,y=y\,.\,x$$

is provable in $\mathscr{A}$. To show this we introduce a model which satisfies the axioms of $\mathscr{A}$ but not theorems (i). Such a model consists of the natural numbers 0, 1, 2, ... combining in the usual way under addition and multiplication together with two extra numbers α and β such that

$$S\alpha=\alpha,\quad S\beta=\beta,\quad \alpha+n=\alpha,\quad \beta+n=\beta,\quad n+\alpha=\beta,\quad n+\beta=\alpha,$$
$$\alpha\,.\,0=0,\quad \beta\,.\,0=0,\quad \alpha\,.\,Sn=\beta,\quad \beta\,.\,Sn=\alpha,\quad n\,.\,\alpha=\alpha,\quad n\,.\,\beta=\beta$$

It is readily verified that this model satisfies the axioms of $\mathscr{A}$ and therefore

satisfies any sentence deduced from $\mathscr{A}$, but the model obviously does not satisfy theorems (i). What is lacking in $\mathscr{A}$ is mathematical induction, and if we added to the axioms of $\mathscr{A}$ all the infinitely many instances of the axiom

$$P(0) \;\&\; (Ax)\{P(x) \rightarrow P(Sx)\} \rightarrow Ax\, P(x)$$

for all arithmetical predicates* P of the system (and an axiom making " = " a transitive relation) then we would be able to prove all the familiar theorems of classical number theory.

In spite of its deficiencies system $\mathscr{A}$ has a remarkable property ; it can be shown that to every *computable* function $f(n)$, whose values may be computed on a machine with limited or unlimited memory, there corresponds a predicate $\phi(x, y)$ such that for any numbers N_x, N_y the sentence

$$\phi(N_x, N_y)$$

is provable or refutable in $\mathscr{A}$ according as y does or does not equal $f(x)$. The importance of system $\mathscr{A}$ lies in the fact that it attains this property with only a finite number of axioms.

We shall show that this fragment of arithmetic is undecidable ; the proof of undecidability will not, however, be confined to system $\mathscr{A}$ but will apply to any consistent extension of $\mathscr{A}$ obtained by introducing additional axioms, for instance mathematical induction.

The proof of undecidability is based on Gödel's famous *arithmetisation of syntax*. Arithmetisation starts by assigning a number (which we take to be odd) to each arithmetical and logical sign. For instance we might be given the signs 0, S, =, +, . the numbers 1, 9, 17, 25, 33, keeping other numbers of the form $8n+1$ for brackets and logical signs ; then individual variables might receive numbers of the form $8n+3$ and predicate variables numbers of the form $8n+5$.

A sentence made up of signs with the numbers

$$n_0, n_1, n_2, \ldots, n_k$$

(in this order) is given the number

$$2^{n_0} \cdot 3^{n_1} \cdot 5^{n_2} \ldots p_k^{n_k}$$

where p_k is the kth odd prime.

Similarly a sequence of sentences (e.g. a proof) with numbers $s_1, s_2, \ldots, s_l$ is given the number

$$2^{s_0} \cdot 3^{s_1} \ldots p_l^{s_l}.$$

Given the number of a sentence, the sentence itself may be found by resolving the sentence number into its prime factors, and in the same way the constituent sentences of a proof may be regained from the number of the proof. Thus we have a code in which every symbol and every message is expressed by a single number.

It is comparatively simple to formulate the arithmetical relationship between the number of a conclusion and the numbers of the premisses from which it is inferred by a given rule of inference, and hence to find the conditions which a number must satisfy to be the number of a proof. Arithmetisation, in fact, as the name suggests, maps the syntax of arithmetic on a part of arithmetic itself, and was used by Kurt Gödel in 1931 to prove that if arithmetic is consistent then this consistency is not provable in arithmetic.

* An arithmetical predicate is a logical combination of equations between terms made up of variables and the signs 0, S, + and . ; e.g. $(Ex)(Ay)\{(x+y) \cdot z = Sz\}$.

In virtue of arithmetisation we may take the hypothesis that arithmetic is decidable to mean that there is a computable function Dn such that $Dn=0$ if and only if n is the number of a valid sentence. If Dn exists there is a predicate $\Delta(x)$, say, in $\mathscr{A}$, such that either $\Delta(N_n)$ or $\neg\Delta(N_n)$ is valid according as Dn is or is not zero.

Denote by Sub n the number of the sentence obtained by substituting the numeral N_n for the free variable in sentence number n; Sub n is obviously computable and so there is a predicate $\Sigma(x, y)$, say, such that

$$\Sigma(N_m, N_n)$$

is valid if and only if $n=\text{Sub } m$.

Let $\mathbf{p}$ be the number of the sentence

$$(Ay)\{\Sigma(x, y)\rightarrow\neg\Delta(y)\}$$

so that Sub $\mathbf{p}$ is the number of the sentence

$$(Ay)\{\Sigma(N_{\mathbf{p}}, y)\rightarrow\neg\Delta(y)\},$$

which we denote by P.

If P is valid in $\mathscr{A}$ then

$$\Sigma(N_{\mathbf{p}}, N_{\text{Sub }\mathbf{p}})\rightarrow\neg\Delta(N_{\text{Sub }\mathbf{p}})$$

is valid; since the antecedent $\Sigma(N_{\mathbf{p}}, N_{\text{Sub }\mathbf{p}})$ is valid, by definition, it follows that if P is valid then $\neg\Delta(N_{\text{Sub }\mathbf{p}})$ is valid. On the other hand, if P is not valid, then $\neg\Delta(N_{\text{Sub }\mathbf{p}})$ is valid, since Sub $\mathbf{p}$ is the number of P, and so

$$\neg\Delta(N_{\text{Sub }\mathbf{p}})$$

is valid in either case, from which it follows that

$$\Sigma(N_{\mathbf{p}}, N_{\text{Sub }\mathbf{p}})\rightarrow\neg\Delta(N_{\text{Sub }\mathbf{p}})$$

is valid.

But if $y\neq N_{\text{Sub }\mathbf{p}}$ then $\Sigma(N_{\mathbf{p}}, y)$ is false and therefore

$$\Sigma(N_{\mathbf{p}}, y)\rightarrow\neg\Delta(y)$$

is valid.

It follows that

$$(Ay)\{\Sigma(N_{\mathbf{p}}, y)\rightarrow\neg\Delta(y)\}$$

is valid, i.e. sentence number Sub $\mathbf{p}$ is valid and so

$$\Delta(N_{\text{Sub }\mathbf{p}})$$

is valid and arithmetic contains a contradiction.

Thus if arithmetic is free from contradiction it is undecidable.

The undecidability of predicate logic is deducible from the undecidability of system $\mathscr{A}$. A necessary preliminary to the derivation is the elimination of the extra-logical predicate and individual symbols and the operators $+$ and $.$ from $\mathscr{A}$. To this end we write the predicate $E(x, y)$ for the equation $x=y$, the predicate $S_0(x, y)$ for the equation $Sx=y$, and the predicates $S_1(x, y, z)$, $P_1(x, y, z)$ for the equations $x+y=z$ and $x\,.\,y=z$ respectively. To eliminate the constant 0 we use axioms 2 and 3 and write $(Az)\rightarrow(x=Sz)$ for the equation $x=0$. Under these transformations axioms 2 and 3 become universally valid, axiom 1 becomes

$$S_0(x, z)\ \&\ S_0(y, z)\rightarrow E(x, y),$$

and the axioms for addition and multiplication are transformed into

$$\{(Az) \rightarrow S_0(z, w)\} \rightarrow S_1(x, w, x)$$
$$\{S_0(y, u) \,\&\, S_0(v, w) \,\&\, S_1(x, y, v)\} \rightarrow S_1(x, u, w)$$

and

$$\{(Az) \rightarrow S_0(z, w)\} \rightarrow P_1(x, w, w)$$
$$S_0(y, u) \,\&\, S_1(v, x, w) \,\&\, P_1(x, y, v) \rightarrow P_1(x, u, w).$$

Let the conjunction of these transforms (of which there are only a finite number) be denoted by F, and consider any sentence $S_{\mathscr{A}}$ in system $\mathscr{A}$. By the given transformations $S_{\mathscr{A}}$ becomes a predicate sentence S say, and $S_{\mathscr{A}}$ is provable in $\mathscr{A}$ if, and only if, S is derivable from F in predicate logic. But by the deduction theorem, S is derivable from F if, and only if,

$$F \rightarrow S$$

is provable in predicate logic, regarding E, S_0, S_1 and P_1 as predicate variables. Hence if predicate logic were decidable then $F \rightarrow S$ would be decidable; if $F \rightarrow S$ is universally valid then since F is provable in $\mathscr{A}$, $S_{\mathscr{A}}$ is provable in $\mathscr{A}$, and if $\neg(F \rightarrow S)$, i.e. $F \,\&\, \neg S$ is satisfiable then $S_{\mathscr{A}}$ is not provable in $\mathscr{A}$. Thus if predicate logic were decidable we should have a decision procedure for the undecidable system $\mathscr{A}$.

R.L.G.

BIBLIOGRAPHICAL NOTES

(The numbers in square brackets refer to the Bibliography)

The Arithmetisation of Syntax was introduced by K. Gödel in [2].

The first proof of the undecidability of predicate logic was given by A. Church in [1].

A decision method for monadic predicate logic was first given by L. Löwenheim in [3].

The truth-table decision method for sentence logic was first obtained by L. Wittgenstein in [9] and E. Post in [4].

A decision method for Arithmetic without multiplication was given by M. Presburger in [5].

The foregoing proof of the essential undecidability of Arithmetic is due to A. Tarski, A. Mostowski and R. M. Robinson and is given in [8].

The decision method for elementary algebra was obtained by A. Tarski and written up by J. C. C. McKinsey in [7], and that for Abelian Groups is given by Wanda Szmielew in [6].

REFERENCES

1. Alonzo Church, "A note on the Entscheidungsproblem," *Journal of Symbolic Logic*, Vol. I (1936), pp. 40–41; pp. 101–102. "An unsolvable problem of elementary number theory," *American Journal of Mathematics*, Vol. 58 (1936), pp. 345–363.

2. Kurt Gödel, "Über formal unentscheidbare Satze der Principia Mathematica und verwandter Systeme I," *Monatshefte für Mathematik und Physik*, Vol. 38 (1931), pp. 173–198.

3. Leopold Löwenheim, "Über Möglichkeiten in Relativkalkül, *Mathematische Annalen*, Vol. 76 (1915), pp. 447–470.

4. Emil L. Post, " Introduction to a general theory of elementary propositions, *American Journal of Mathematics*, Vol. 43 (1921), pp. 163–185.

5. M. Presburger, " Über die Vollständichkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt, " *Comptes-rendus du I Congrès des Mathématiciens des Pays Slaves* (Warszawa, 1929), pp. 92–101 and 395.

6. Wanda Szmielew, " Decision problem in group theory," *Proceeding Xth International Congress of Philosophy* (Amsterdam, 1948), fasc. 2, pp. 763–6.

7. Alfred Tarski, *A decision method for elementary algebra and geometry*, California, 1948.

8. Alfred Tarski, Andrzej Mostowski and Raphael M. Robinson, *Undecidable Theories*, Amsterdam, 1953.

9. Ludwig Wittgenstein, *Tractatus logico-philosophicus*, London, 1922.

ARCHIMEDEAN SCREWS

By H. G. ApSimon

1. A *helicoid* is the surface generated by a curve which is simultaneously rotated about a fixed axis and translated in the direction of the axis with a velocity proportional to the angular velocity of rotation.[1] An Archimedean Screw[2] is strictly a finite helicoid in which the generating curve is closed (and normally circular); but screws, helical in shape, for which the generating curve is a simple finite arc can be—and are—used in practice as Archimedean Screws, and are considerably easier to manufacture than tubular ones.

The construction is still further facilitated if the helicoid can be made out of flat sheets—that is, if it is developable—and accordingly I investigate only developable helicoids.

2. *The equation of the developable helicoid.*

The general helicoid may, by a suitable choice of axes, be taken as

$$\left.\begin{aligned} x &= u\cos v, \\ y &= u\sin v, \\ z &= f(u)+cv. \end{aligned}\right\} \qquad \text{.....(1)}$$

Using the methods and notation of differential geometry, it is found that for this surface

$$HL = uf''(u), \qquad HM = -c, \qquad HN = u^2 f'(u);$$

from which

$$H^2T^2 = u^2 f'(u)f''(u) - c^2.$$

Now the necessary and sufficient condition for the surface to be developable is

$$T^2/H^2 = 0,$$

and as we can immediately verify for this surface that $H > 0$, we have that the surface is developable if and only if

$$u^2 f'(u)f''(u) = c^2,$$

which on integration gives

$$f(u) = c\sqrt{\{u^2/a^2 - 1\}} - c\sec^{-1}(u/a)\ ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where a is an arbitrary constant arising from the integration (the other constant of integration becomes zero on a suitable choice of origin.)

3. *The generation of the developable helicoid.*

As any developable surface is ruled, helical symmetry suggests that the developable helicoid is generated by the tangents to a helix of the same pitch, and accordingly we consider the helix

$$\mathbf{r} = (a\cos v,\ a\sin v,\ cv)$$

from which

$$\mathbf{t} = (a^2 + c^2)^{-\frac{1}{2}}(-a\sin v,\ a\cos v,\ c)$$

and

$$\mathbf{n} = (-\cos v,\ -\sin v,\ 0),$$

so that

$$\mathbf{t} \wedge \mathbf{n} = (c\sin v,\ -c\cos v,\ a).$$

Since the osculating plane is given by

$$[\mathbf{R} - \mathbf{r},\ \mathbf{t},\ \mathbf{n}] = 0,$$

the osculating planes of the helix form the family

$$F(x, y, z, v) \equiv cx\sin v - cy\cos v + az - cav = 0.$$

The envelope of this family is given by eliminating v between this equation and

$$\frac{\partial F}{\partial v} \equiv c(x\cos v + y\sin v - a) = 0\ ;$$

leading to

$$z = c\sqrt{\{(x^2 + y^2)/a^2 - 1\}} - c\sec^{-1}\{a^{-1}\sqrt{(x^2 + y^2)}\} + c\tan^{-1}(y/x)$$

which, as is easily verified, is the helicoid defined by (1) and (2).

Hence the developable helicoid of pitch $2\pi c$ and minimal radius a is generated by the tangents to the helix of pitch $2\pi c$ and radius a (and enveloped by the one-parameter family of planes which are the osculating planes of this helix).

4. *The construction of a developable helicoid.*

Consider a plane lamina formed by the arc C_1 of a circle of radius r_1 subtending an angle α at the centre of the circle, the curve C_2 (an arc of a concentric circle of radius $r_2 = \sqrt{(L^2 + r_1^2)}$) traced out by the tangents to C_1 of length L, and the bounding tangents of C_1.

It is possible to deform C_1 into part of a helix of pitch $2\pi c$ having one complete turn and radius a, where

$$2\pi\sqrt{(a^2 + c^2)} = \alpha r_1.$$

It is, further, necessary that the curvature of C_1 remain unaltered by this deformation ; the curvature of the helix is given by

$$t' = \kappa n,$$

so

$$\kappa = a(a^2 + c^2)^{-1}.$$

Then since the curvature of C_1 is r_1^{-1}, we have

$$r_1 = (a^2 + c^2)/a ;$$

and consequently

$$\alpha/2\pi = a/\sqrt{(a^2 + c^2)}.$$

Then the tangents to C_1 become the tangents to the helix and the lamina becomes the surface generated by these tangents; i.e. the developable helicoid.

The tangents being segments of length L, the part of the helicoid obtained is bounded by a cylinder, coaxial with the helix, of radius R such that

$$2\pi\sqrt{(R^2 + c^2)} = \alpha r_2.$$

5. Hence the plane lamina necessary to make one turn of a developable helicoid of internal (minimal) radius a, pitch $2\pi c$, and external radius R is that part of an annulus of internal and external radii

$$\frac{a^2 + c^2}{a} \quad \frac{\sqrt{(a + c^2)}\sqrt{(R^2 + c^2)}}{a}$$

which is bounded by tangents to the inner circle whose points of contact subtend at the centre an angle

$$\frac{2a\pi}{\sqrt{(a^2 + c^2)}}.$$

H. G. ApS.

REFERENCES

1. C. E. Weatherburn, *Differential Geometry of Three Dimensions*, Vol. 1, (1st edition, Cambridge, 1927) p. 65. (The notation used in this note is that of this work.)

2. See, for instance, W. W. R. Ball, *A Short History of Mathematics*, (4th edition, Macmillan, 1908) p. 65, for a brief description of the applications of Archimedean Screws.

1875. [Ernest Barker, nearing his twelfth birthday, was put to work with another boy in the village school, with the intent that he should act as pace-maker to the latter in an attempt for a scholarship at Manchester Grammar School.] " My chief memory of our work is . . . of Euclid . . . . There was a clinchingness about Euclid's proofs which impressed me deeply. Here was something different from the airy world of Scott—something with a precision, and a sort of click as the pieces fell home, that stirred my mind and stimulated me to thought. I doubt whether my knowledge of Euclid was anything more than mnemonic; and yet I remember the Headmaster saying that I understood the propositions, and could argue them out for myself. I think he was optimistic." [The plan failed. It was Barker who won the scholarship.]—Sir Ernest Barker, *Age and Youth* (Oxford University Press, 1953), p. 254. [Per Mr. J. T. Combridge.]

MATHEMATICAL NOTES

2661. Sur une équation linéaire aux dérivées partielles à coefficients constants.

1. Considérons l'équation

$$a\frac{\partial^2 z}{\partial x^2}+2b\frac{\partial^2 z}{\partial x\partial y}+c\frac{\partial^2 z}{\partial y^2}=0, \quad (a, b, c \text{ constantes}), \tag{1}$$

et son équation caractéristique

$$a+2br+cr^2=0, \quad (r_1, r_2 \text{ ses racines}). \tag{2}$$

Si $r_2=r_1$, on indique {*cf.*, par exemple, [1], t. I, p. 145 et [2], p. 463} un procédé spécial fournissant la solution générale de l'équation (1), procédé qui n'utilise pas la solution suivante de (1) :

$$z=f(x+r_1y)+g(x+r_2y), \quad (r_1\neq r_2), \tag{3}$$

(f, g fonctions arbitraires, supposées dérivables, en arguments indiqués).

En modifiant convenablement le procédé de *D'Alembert*, relatif au problème analogue pour les équations linéaires aux derivées ordinaires, {*cf.* [1], t. II, p. 461}, nous allons montrer, dans cette Note, comment on déduit la solution générale de (1), dans le cas où $r_2=r_1$ en partant de la solution (3), avec $r_1\neq r_2$. En outre, nous allons étendre le procédé en question aux équations linéaires de l'ordre supérieur à deux.

2. Si $r_1\neq r_2$, l'équation (1) a, comme solution générale,

$$z=f(x+r_1y)+g(x+r_2y).$$

Par suite, la fonction

$$\frac{g(x+r_2y)-g(x+r_1y)}{r_2-r_1}, \quad r_2\neq r_1,$$

est aussi une solution de (1).

Par un raisonnement, analogue à celui suivi par *Goursat* {*cf.* [1], t. II, p. 461} et qui est relatif aux équations linéaires aux dérivées ordinaires, on voit que la fonction.

$$\lim_{r_2\to r_1}\frac{g(x+r_2y)-g(x+r_1y)}{r_2-r_1}$$

ou bien

$$yg'(x+r_1y)$$

est aussi une solution de (1), correspondant au cas $r_1=r_2$. Donc, la solution générale de (1) est

$$z=f(x+r_1y)+yg(x+r_1y), \quad (r_2=r_1),$$

f et g étant des fonctions arbitraires en arguments mis en évidence.

3. Considérons à présent l'équation linéaire du troisième ordre

$$a\frac{\partial^3 z}{\partial x^3}+3b\frac{\partial^3 z}{\partial x^2\partial y}+3c\frac{\partial^2 z}{\partial x\partial y^2}+d\frac{\partial^3 z}{\partial y^3}=0 \quad (a, b, c, d=\text{const.}) \tag{4}$$

à laquelle correspond l'équation caractéristique

$$a+3br+3cr^2+dr^3=0.$$

Si $r_1 \neq r_2 \neq r_3$, la solution générale de (4) est la fonction

$$z = f_1(x + r_1 y) + f_2(x + r_2 y) + f_3(x + r_3 y),$$

(f_1, f_2, f_3 fonctions arbitraires en arguments indiqués).

Dans le cas où $r_1 = r_2 \neq r_3$, la fonction

$$z = f_1(x + r_1 y) + y f_2(x + r_1 y) + f_3(x + r_3 y)$$

est la solution générale de (4).

La fonction

$$\frac{f_3(x + r_3 y) - f_3(x + r_1 y) - (r_3 - r_1) y f_3'(x + r_1 y)}{(r_3 - r_1)^2}$$

est aussi une solution de (4), si $r_1 = r_2 \neq r_3$.

La fonction

$$\lim_{r_3 \to r_1} \frac{f_3(x + r_3 y) - f_3(x + r_1 y) - (r_3 - r_1) y f_3'(x + r_1 y)}{(r_3 - r_1)^2},$$

ou bien

$$\frac{y^2}{2} f_3'(x + r_1 y)$$

est aussi une solution de (4), correspondant au cas $r_1 = r_2 = r_3$.

Par suite, si $r_1 = r_2 = r_3$, l'équation (4) a, comme solution generale, la fonction

$$z = f_1(x + r_1 y) + y f_2(x + r_1 y) + y^2 f_3(x + r_1 y).$$

3. Passons enfin au cas plus général ; considérons l'équation linéaire de l'ordre n, à coefficients constants, de la forme

$$\left(A_0 \frac{\partial^n}{\partial x^n} + A_1 \frac{\partial^n}{\partial x^{n-1} \partial y} + \ldots + A_n \frac{\partial^n}{\partial y^n}\right) z = 0 \tag{5}$$

avec A_k constantes quelconques.

Supposons que l'équation caractéristique attachée à l'équation (5) ait les p, ($p \leqslant n$), racines égales à r, ($r_1 = r_2 = r_3 = \ldots = r_p \equiv r$). Alors la solution générale de (5) est

$$z = f_1(x + ry) + y f_2(x + ry) + \ldots + y^{p-1} f_p(x + ry) + \{\ldots\},$$

en désignant par $\{\ldots\}$ une fonction qui doit être formée en partant des autres racines (dont le nombre est $n - p$) de l'équation caractéristique attachée à (5).

Nous démontrerons ceci dans ce qui suit.

L'un après l'autre nous allons considérer les cas suivants :

$$\begin{aligned}
&r_1 \neq r_2 \neq r_3 \neq \ldots \neq r_p ; \\
&r_1 = r_2 \neq r_3 \neq \ldots \neq r_p ; \\
&r_1 = r_2 = r_3 \neq r_4 \neq \ldots \neq r_p ; \\
&\qquad\vdots \\
&r_1 = r_2 = r_3 = \ldots = r_p,
\end{aligned}$$

et nous allons supposer que l'assertion indiquée soit vraie pour

$$r \equiv r_1 = r_2 = \ldots = r_{p-1} \neq r_p,$$

c'est-à-dire que l'équation (5) ait pour la solution générale la fonction qui suit

$$z = \sum_{k=0}^{p-2} y^k f_{k+1}(x + ry) + f_p(x + r_p y) + \{\ldots\}.$$

En suivant le procédé indiqué dans le § 3 de cette Note, formons ensuite l'expression.

$$\lim_{r_p \to r} P/(r_p - r)^{p-1}, \tag{6}$$

avec

$$P \equiv f_p(x + r_p y) - f_p(x + ry) - (r_p - r)y f_p'(x + ry) - \frac{(r_p - r)^2}{2!} y^2 f_p''(x + ry) - \ldots - \frac{(r_p - r)^{p-2}}{(p-2)!} y^{p-2} f_p^{(p-2)}(x + ry).$$

Après application répétée de la règle de l'*Hôspital*, on obtient

$$\lim_{r_p \to r} P/(r_p - r)^{p-1} = \frac{1}{(p-1)!} y^{p-1} f_p^{(p-1)}(x + ry),$$

ce qui complète notre assertion mentionnée plus haut.

4. Les faits précédemment indiqués sont élémentaires, mais nous ne les avons pas rencontrés dans la littérature mathématique. Ils peuvent, au moins, présenter quelque intérêt dans l'enseignement des équations aux dérivées partielles.

D. S. MITRINOVITCH
Belgrade, Yougoslavia.

INDEX BIBLIOGRAPHIQUE

1. E. Goursat, *Cours d'Analyse mathématique*, t. I, cinquième édition, Paris, 1927 ; t. II, cinquième édition, Paris, 1929.

2. A. R. Forsyth and W. Jacobsthal, *Lehrbuch der Differentialgleichungen, zweite Auflage*, Braunschweig, 1912.

2662. Algebraic proof that $\int_a^b x^r\,dx = (b^{r+1} - a^{r+1})/(r+1)$ (r integral).

Fermat's pre-calculus proof (*v.* Goursat, *Mathematical Analysis*, § 66) that the area enclosed by $y = Ax^r$, $y = 0$, $x = a$, $x = b$ is $A(b^{r+1} - a^{r+1})/(r+1)$ can be used as an interesting verification of a result later established by integration. It appears to the writer even simpler to adapt the easily proved theorem (C. Smith, *Algebra*) that (§ 318), if $u_s = \prod_{p=1}^{r} [a + (p + s - 2)k]$, $v_s = u_s[a + (r + s - 1)k]$,

then $k \sum_{s=1}^{n} u_s = (v_n - v_0)/(r+1)$ and (§ 319), if $u_s' = 1/u_s$, $v_s' = [a + (s-1)k]u_s'$,

then $k \sum_{s=1}^{n} u_s' = (v_0' - v_n')/(r-1)$.

For, put $b = a + nk$, $\delta x = k = (b-a)/n \to 0$ as $n \to \infty$. Then $u_s \to (a + sk)^r = x_s^r$ (say), so that $v_s \to x_s^{r+1}$, $v_0 \to a^{r+1}$, $v_n \to b^{r+1}$. Hence

$$\lim_{\delta x \to 0,\, n \to \infty} \sum_{s=1}^{n} x_s^r \delta x = (b^{r+1} - a^{r+1})/(r+1).$$

Again, as $\lim_{k \to 0} u_s' = (a + sk)^{-r} = x_s^{-r}$, then $\lim_{k \to 0} v_s' = x_s^{-r+1}$, $v_0' \to a^{-r+1}$, $v_n' \to b^{-r+1}$

$$\therefore \lim_{\delta x \to 0,\, n \to \infty} \sum_{s=1}^{n} x_s^{-r} \delta x = -(b^{-(r-1)} - a^{-(r-1)})/(r-1).$$

34 Stonehill Road, S.W.14 A. J. CARR

2663. On the arithmetico-geometric series.

Taking n and r as positive integers and denoting, as usual, the factorial product $n(n-1)\ldots(n-r+1)$ by n_r, the general Arithmetico-Geometric Series is ${}_rS_{n+1}(x)$, defined by

$$ {}_rS_{n+1}(x) \equiv 1 + {}_rC_1 x + {}_{r+1}C_2 x^2 + \ldots + {}_{n+r-1}C_n x^n \quad \ldots\ldots(1) $$

$$ \therefore\ {}_1S_{n+r}(x) = 1 + x + x^2 + \ldots + x^{r-1} + x^r + \ldots + x^{n+r-1} \quad \ldots\ldots(2) $$

Differentiating (2) $(r-1)$ times and denoting $d^p {}_1S_{n+r}(x)/dx^p$ by ${}_1S^{(p)}_{n+r}$, with ${}_1S^{(0)}_{n+r}(x) \equiv {}_1S_{n+r}(x)$, we have

$$ {}_1S^{(r-1)}_{n+r}(x) = (r-1)_{r-1} + r_{r-1}x + (r+1)_{r-1}x^2 + \ldots + (n+r-1)_{r-1}x^n $$

or
$$ {}_1S^{(r-1)}_{n+r}(x)/(r-1)! = 1 + {}_rC_1 x + {}_{r+1}C_2 x^2 + \ldots + {}_{n+r-1}C_n x^n $$

$$ = {}_rS_{n+1}(x) \quad [\text{by (1)}] \quad \ldots\ldots(3) $$

But, as (2) is a G.P., we can write

$$ (1-x)\,{}_1S_{n+r}(x) = 1 - x^{n+r} \quad \ldots\ldots(4) $$

and, differentiating (4) p times, we derive, by Leibniz' Theorem,

$$ (1-x)\,{}_1S^{(p)}_{n+r}(x) - p\,{}_1S^{(p-1)}_{n+r}(x) = -(n+r)_p x^{n+r-p} \quad \ldots\ldots(5) $$

i.e.
$$ \frac{(1-x)^p\,{}_1S^{(p)}_{n+r}(x)}{p!} - \frac{(1-x)^{p-1}\,{}_1S^{(p-1)}_{n+r}(x)}{(p-1)!} = -{}_{n+r}C_p (1-x)^{p-1} x^{n+r-p} \quad \ldots\ldots(6) $$

Adding all equations of the type (6), from $p=1$ to $p=r-1$, we get

$$ \frac{(1-x)^{r-1}\,{}_1S^{(r-1)}_{n+r}(x)}{(r-1)!} - {}_1S^{(0)}_{n+r}(x) = -\sum_{p=1}^{r-1} {}_{n+r}C_p (1-x)^{p-1} x^{n+r-p} \quad \ldots\ldots(7) $$

$$ \therefore\ {}_rS_{n+1}(x) = {}_1S^{(r-1)}_{n+r}(x)/(r-1)! \quad [\text{by (3)}] $$

$$ = \frac{1-x^{n+r}}{(1-x)^r} - \sum_{p=1}^{r-1} {}_{n+r}C_p \frac{x^{n+r-p}}{(1-x)^{r-p}} \quad [\text{by (7) and (4)}] $$

$$ = \frac{1}{(1-x)^r} - x^n \sum_{p=1}^{r-1} {}_{n+r}C_p \left(\frac{x}{1-x}\right)^{r-p} \quad \ldots\ldots(8) $$

Obviously, if $|x|<1$, then, as $n \to \infty$, the second term on the right of (8) tends to zero and so we get the Binomial Theorem for a negative integral index. Also, comparing (1), (8), we get, for any n,

$$ \sum_{p=n+1}^{\infty} {}_{p+r}C_p x^p = x^n \sum_{p=0}^{r-1} {}_{n+r}C_p \left(\frac{x}{1-x}\right)^{r-p} \quad (|x|<1). $$

It is interesting, moreover, to note that if $x=1$, $p=r$, then (5) gives

$$ {}_1S^{(r-1)}_{n+r}(1)/(r-1)! = {}_{n+r}C_r = {}_{n+r}C_n \quad \ldots\ldots(9) $$

As n is a positive integer, the combination of (9) with (3) for $x=1$, viz.

$$ {}_1S^{(r-1)}_{n+r}(1)/(r-1)! = 1 + {}_rC_1 + {}_{r+1}C_2 + \ldots + {}_{n+r-1}C_n \quad \ldots\ldots(10) $$

yields the well-known result

$$ {}_{r+1}H_n = {}_rH_0 + {}_rH_1 + {}_rH_2 + \ldots + {}_rH_n, $$

where ${}_rH_n$ is the sum of the homogeneous products of the nth degree of r quantities.

The following alternative method of attack gives a result at first sight rather different from (8). Let y be less than the smaller of the two numbers unity, x^{-1}. Then by expanding $(1-xy)^{-r}(1-y)^{-1}$ in positive powers of y, it is seen that the coefficient of y^n is ${}_rS_{n+1}(x)$. But

$$\begin{aligned}\frac{1}{(1-xy)^r(1-y)} &= \frac{-x}{(1-xy)^r}\frac{1}{1-x-(1-xy)} \\ &= \frac{-x}{(1-xy)^r}\left\{\sum_{s=0}^{r-1}\frac{(1-xy)^s}{(1-x)^{s+1}} - \frac{1}{x(1-y)}\frac{(1-xy)^r}{(1-x)^r}\right\} \\ &= \frac{1}{1-y}\frac{1}{(1-x)^r} - x\sum_{s=0}^{r-1}\frac{(1-xy)^{-(r-s)}}{(1-x)^{s+1}} \\ &= (1-x)^{-r}\sum_{n=0}^{\infty} y^n - x\sum_{s=0}^{r-1}(1-x)^{-s-1}\sum_{n=0}^{\infty}{}_{r-s+n-1}C_n x^n y^n.\end{aligned}$$

Thus
$${}_rS_{n+1}(x) = (1-x)^{-r} - x^{n+1}\sum_{s=0}^{r-1}{}_{n+r-s-1}C_n(1-x)^{-s-1} \quad \text{.....(11)}$$

To show that (8) and (11) are the same, we have

$$\begin{aligned}\sum_{p=0}^{r-1}{}_{n+r}C_p\left(\frac{x}{1-x}\right)^{r-p} &= \frac{x}{1-x}\sum_{p=0}^{r-1}{}_{n+r}C_{r-1-p}\left(\frac{1}{1-x}-1\right)^p \\ &= x(1-x)^{-1}\sum_{p=0}^{r-1}{}_{n+r}C_{r-1-p}\sum_{s=0}^{p}{}_pC_s(-)^{p-s}(1-x)^{-s} \\ &= x\sum_{s=0}^{r-1}(1-x)^{-s-1}\sum_{p=s}^{r-1}(-)^{p-s}{}_{n+r}C_{r-p-1}\cdot{}_pC_s \quad \text{.....(12)}\end{aligned}$$

Thus, by comparing (8), (11), (12), we find

$$\sum_{p=s}^{r-1}(-)^{p-s}{}_{n+r}C_{r-p-1}\cdot{}_pC_s = {}_{n+r-s-1}C_n \quad \text{.....(13)}$$

Equivalently, we may aver that, by slightly adjusting the letters,

$$\sum_{k=0}^{l}(-)^k{}_{n+s+l+1}C_{l-k}\cdot{}_{k+s}C_s = {}_{n+l}C_l \quad \text{.....(14)}$$

for l, n, s being any three positive numbers whatever. I have not succeeded in verifying (14) independently except in several numerical cases in which l, n, s were chosen quite at random. In particular, for $n=0$, (14) gives

$$\sum_{k=0}^{l}(-)^k{}_{s+l+1}C_{l-k}\cdot{}_{k+s}C_s = 1.$$

To find $\Sigma_n^{(r)}(x) \equiv \sum_{p=1}^{n} p^r x^r$.

An immediate deduction from either (8) or (11) is the value of $\Sigma_n^{(r)}(x)$. In the notation of the Calculus of Finite Differences with the unit difference unity itself, we have, for $x \neq 1$,

$$\begin{aligned}\Sigma_n^{(r)}(x) &= \sum_{p=1}^{n} x^p(1+\Delta)^p 0^r = \sum_{p=1}^{n} x^p \sum_{s=1}^{p}{}_pC_s\Delta^s 0^r \quad (\text{as } 0^r = 0) \\ &= \sum_{s=1}^{m}\Delta^s 0^r \sum_{p=s}^{n}{}_pC_{p-s}x^p \quad \text{.....(15)}\end{aligned}$$

(with $m=n$ for $r \geqslant n$; $m=r$ for $r<n$ as $\Delta^{r+1}0^r=0$)

$$= \sum_{s=1}^{m} \Delta^s 0^r \left\{ \frac{x^s}{(1-x)^{s+1}} - x^n \sum_{q=0}^{s} {}_{n+1}C_q \left(\frac{x}{1-x} \right)^{s+1-q} \right\} \quad \text{[by (8)]} \quad \ldots\ldots\ldots(16)$$

or $$= \sum_{s=1}^{m} \Delta^s 0^r \{ x^s (1-x)^{-s-1} - x^{n+1} \sum_{q=0}^{s} {}_{n-q}C_{n-s} (1-x)^{-q-1} \} \quad \text{[by (11)]} \quad \ldots(17)$$

where

$$\Delta^s 0^r = \sum_{p=0}^{s-1} (-)^p {}_sC_p (s-p)^r.$$

For $r=2$, putting $\Delta 0^2=1$, $\Delta^2 0^2 = 2^2 - 2 \cdot 1 = 2$ in either (16) or (17), we get $\Sigma_n^{(2)}(x) = (1-x)^{-3}[x+x^2-(n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}]$
(cf. Chrystal, *Text-book of Algebra*, Chap. XX, § 14). If $x=1$, we use (15), giving

$$\sum_{p=1}^{n} p^r = \Sigma_n^{(r)}(1) = \sum_{s=1}^{r} \Delta^s 0^r \sum_{p=s}^{n} {}_pC_{p-s} = \sum_{s=1}^{m} \Delta^s 0^r {}_{n+1}C_{s+1} \text{ by (9), (10).}$$

A. J. CARR

2664. Equivalent problems.

The following geometrical problems are essentially equivalent and may not be generally known.

(i) Given a parallelogram $ABCD$ and four parallel lines through A, B, C, D not in the same plane, to draw a square $A'B'C'D'$ whose vertices are respectively on the four lines.

(ii) To project (orthogonally) an isosceles right-angled triangle into a triangle of given species.

(iii) Given three non-coplanar parallel lines, to draw from a point O on one of them, lines OA, OB to the others so that OA is equal and at right-angles to OB.

Here is a proof of the existence of a solution of (iii). Let the lines be p_O, p_A, p_B. Through O draw a plane π perpendicular to the three lines and cutting them in O, A_1, B_1. Suppose that the angle A_1OB_1 is acute. Let A_2 be another point on p_A and B another point on p_B. If A_2 is kept fixed and B moves from B_1 to ∞ on the opposite side of π to A_2, the angle A_2OB increases from an acute value (when B is at B_1) to an obtuse value (when B is at ∞). Hence there is just one point B_2 on p_B such that A_2OB_2 is a right angle. This is so for every point A on p_A and, as A moves from A_1 to ∞, B moves from ∞ to B_1. Thus the ratio OA/OB increases from 0 to ∞ and there is just one position when $OA/OB=1$ (or indeed any other positive number, showing that the word " isosceles " in (ii) is redundant).

If the angle A_1OB_1 is obtuse, work with the line p_B' parallel to p_B and an equal distance the other side of p_A.

I have not found an easy geometrical *construction*, but it must be possible as the problem is algebraically equivalent to solving simultaneously $xy=$constant, $x^2-y^2=$constant.

University of Southampton. F. G. MAUNSELL

2665. " A pretty series " (Notes 2419 and 2559).

Professor G. N. Watson, after proving $\sum_1^\infty {}^{3r+1}C_r(\frac{2}{27})^r=\frac{1}{2}$ by Kummer's quadratic transformation, Gamma functions and the Gaussian formula for $F(a, b\,;\, c\,;\, 1)$, expressed a hope that a less " high-brow " proof would be forthcoming of this summation, originally given without proof by Mr. D. F. Ferguson.

The following proof, except for some modifications suggested to me by Mr. Ferguson himself, is due to Mr. G. H. Stevens, of Wolstanton Grammar School. It applies to all positive values of x for which the series converges, that is, $0<x<\frac{4}{27}$. Also it depends on nothing beyond School Algebra.

Let $z\equiv x^{\frac{1}{3}}$, and $z<\dfrac{y}{y^3+1}$.

Let $E\equiv\dfrac{1}{1-z\left(\frac{1}{y}+y^2\right)}$

$$\equiv 1+z^1\left(\frac{1}{y}+y^2\right)^1+z^2\left(\frac{1}{y}+y^2\right)^2+z^3\left(\frac{1}{y}+y^2\right)^3$$
$$+z^4\left(\frac{1}{y}+y^2\right)^4+z^5\left(\frac{1}{y}+y^2\right)^5+z^6\left(\frac{1}{y}+y^2\right)^6+z^7\left(\frac{1}{y}+y^2\right)^7+\ \ldots\ \ldots\ldots(1)$$

Coefficient of $\dfrac{1}{y}$ in $(1)=z^1+{}^4C_1z^4+{}^7C_2z^7+\ \ldots$

$$=x^{\frac{1}{3}}(1+{}^4C_1x^1+{}^7C_2x^2+\ \ldots)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Also $$E\equiv\frac{-y/z}{y^3-\frac{y}{z}+1},$$

x being positive, the condition for $y^3-\dfrac{y}{z}+1=0$ to have three real roots is the same as for (2) to converge, $x<\frac{4}{27}$.

One of these roots $(-\alpha)$ must be negative, and two positive (β and γ).

Let $\beta>\gamma$, $\alpha=\beta+\gamma$, $\alpha\beta\gamma=1$.

Take y so that $\beta>|\,y\,|>\gamma$.

Then $E\equiv\dfrac{-y/z}{(y+\alpha)(y-\beta)(y-\gamma)}\equiv\dfrac{y}{z\,.\,\alpha\left(1+\frac{y}{\alpha}\right)\beta\left(1-\frac{y}{\beta}\right)y\left(1-\frac{\gamma}{y}\right)}$

$$\equiv\frac{1}{\alpha\beta z}\left(1+\frac{y}{\alpha}\right)^{-1}\left(1-\frac{y}{\beta}\right)^{-1}\left(1-\frac{\gamma}{y}\right)^{-1}$$
$$\equiv\frac{1}{\alpha\beta z}\left(1-\frac{y^1}{\alpha^1}+\frac{y^2}{\alpha^2}-\frac{y^3}{\alpha^3}+\ \ldots\right)\left(1+\frac{y^1}{\beta^1}+\frac{y^2}{\beta^2}+\frac{y^3}{\beta^3}+\ \ldots\right)$$
$$\times\left(1+\frac{\gamma^1}{y^1}+\frac{\gamma^2}{y^2}+\frac{\gamma^3}{y^3}+\ \ldots\right)\ \ldots\ldots(3)$$

Coefficient of $\dfrac{1}{y}$ in (3)

$$=\frac{1}{\alpha\beta z}\left\{\gamma^1+\gamma^2\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)+\gamma^3\left(\frac{1}{\beta^2}-\frac{1}{\alpha\beta}+\frac{1}{\alpha^2}\right)+\ \ldots\right\}$$

$$= \frac{\gamma}{\alpha\beta z\left(\frac{1}{\beta}+\frac{1}{\alpha}\right)}\left\{\left(\frac{1}{\beta}+\frac{1}{\alpha}\right)+\gamma^1\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)+\gamma^2\left(\frac{1}{\beta^3}+\frac{1}{\alpha^3}\right)+\gamma^3\left(\frac{1}{\beta^4}-\frac{1}{\alpha^4}\right)+\ \ldots\right\}$$

$$= \frac{\gamma}{z(\alpha+\beta)}\left\{\left(\frac{1}{\beta^1}+\frac{\gamma^1}{\beta^2}+\frac{\gamma^2}{\beta^3}+\ \ldots\right)+\left(\frac{1}{\alpha^1}-\frac{\gamma^1}{\alpha^2}+\frac{\gamma^2}{\alpha^3}-\ \ldots\right)\right\}$$

$$= \frac{\gamma}{z(\alpha+\beta)}\left\{\frac{1}{\beta-\gamma}+\frac{1}{\alpha+\gamma}\right\} = \frac{\gamma}{z(\alpha+\gamma)(\beta-\gamma)} \quad \ldots\ldots(4)$$

Equating (2) and (4), we see that the sum of Ferguson's series is

$$\frac{x^{-\frac{2}{3}}\gamma}{(\alpha+\gamma)(\beta-\gamma)} - 1. \quad \ldots\ldots(5)$$

This is the general solution; but it can be put into a more manageable form by using an auxiliary positive acute angle $\phi = \frac{1}{3}\sin^{-1}(1\frac{1}{2}\sqrt{3x})$.

Then $$x = \tfrac{4}{27}\sin^2 3\phi$$

and $$\alpha = \frac{\sin(60^\circ+\phi)}{\cos 30^\circ}x^{-\frac{1}{6}}$$

$$\beta = \frac{\sin(60^\circ-\phi)}{\cos 30^\circ}x^{-\frac{1}{6}}$$

$$\gamma = \frac{\sin\phi}{\cos 30^\circ}x^{-\frac{1}{6}}$$

Hence $$\alpha+\gamma = 2x^{-\frac{1}{6}}\sin(30^\circ+\phi),$$

$$\beta-\gamma = 2x^{-\frac{1}{6}}\sin(30^\circ-\phi),$$

and so $$(\alpha+\gamma)(\beta-\gamma) = 4x^{-\frac{1}{3}}\sin(30^\circ+\phi)\sin(30^\circ-\phi)$$

$$= 2x^{-\frac{1}{3}}(\cos 2\phi - \cos 60^\circ) = x^{-\frac{1}{3}}(4\cos^2\phi - 3)$$

$$= \frac{x^{-\frac{1}{3}}\cos 3\phi}{\cos\phi} \quad \ldots\ldots(6)$$

$\therefore$ From (5) and (6) the sum of the series is one less than

$$\frac{x^{-\frac{1}{2}}\sin\phi\cos\phi}{\cos 30^\circ\cos 3\phi} = \frac{3\sqrt{3}}{2\sin 3\phi}\cdot\frac{\sin\phi\cos\phi}{\cos 30^\circ\cos 3\phi} = \frac{3\sin 2\phi}{\sin 6\phi} \quad \ldots\ldots(7)$$

In the special case $x = \frac{2}{27}$, $\phi = 15^\circ$, and the sum is $\frac{1}{2}$.

Grist Hill Farmhouse, Shamley Green, Guildford. W. HOPE-JONES

Editorial Note. A somewhat similar solution was received from Mr. A. G. D. Watson. Mr. D. G. Tahta submitted a solution by R. Coles, his pupil at St. Albans School, and the same solution was sent in by L. Carlitz, Duke University, North Carolina.

2666. An iterative construction for the trisection of a given angle.

Summary. Most Euclidean constructions for the trisection of a given angle depend on the construction and drawing of a curve. Slow iterative procedures of a naive kind are easy to concoct, but one cannot call to mind any which converge so rapidly that only one, or at most two, iterations are required.

The construction given below is an iteration, the second or at any rate third step of which seems able to give the trisection correct to the accuracy associated with any drawing. An empirical variant of it gives even more rapid convergence.

Construction. We refer to the figure, in which AOB is the angle to be trisected. $\widehat{OAB} = \widehat{EBC} = 90°$, and CDB is an arc of a circle centre O. Choose a point R_1 as near as possible to the point of trisection of the arc CDB next to C. With centre B, radius BR_1, mark Q_1 on BE. With centre A, radius AQ_1, mark P_1 on BC produced. With centre C, radius CP_1, mark S_1 on arc CDB.

Bisect the acute angle S_1BR_1 to cut the arc CDB at R_2. The iteration then proceeds to construct S_2 and so on. The points R_n, S_n converge to a common limit point, T, and $\widehat{TBC}$ is one-third and $\widehat{TCB}$ two-thirds of the angle $\widehat{AOB}$.

The rapidity of the convergence naturally depends on the choice of R_1; even if a very bad choice is made, as in the figure, it has been found that R_4 is usually correct to the usual accuracy of a drawing.

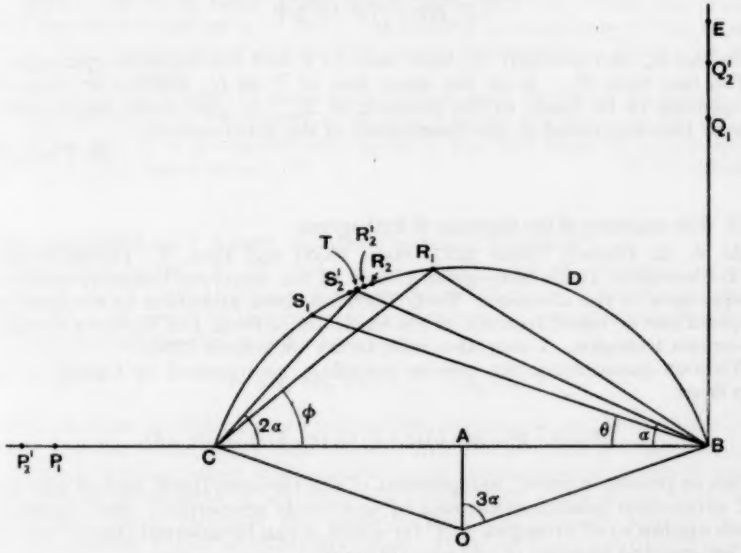


FIG. 1.

The construction is a strict iteration and the proof of its convergence to the correct limit is given later. For practical purposes, however, it is convenient to replace R_2 by a point R_2' which is rather further from R_1 than is R_2; the value of this procedure is indicated in the proof. This technique is illustrated in the figure in which the limit point, T, was taken at the mid-point of $R_2'S_2'$; no greater accuracy could be achieved by following the exact iteration.

Proof.

Let $BC = 1$, $\widehat{AOB} = 3\alpha$, $\widehat{TCB} = 2\alpha$, $\widehat{TBC} = \alpha$, $\widehat{R_1CB} = \phi$, $\widehat{S_1BC} = \theta$, as shewn.

Then $S_1C = \sin\theta \,.\, \operatorname{cosec} 3\alpha$; $AP_1 = \frac{1}{2} + \sin\theta \,.\, \operatorname{cosec} 3\alpha = AQ_1$.

Thus $Q_1B^2 = (\frac{1}{2} + \sin\theta \,.\, \operatorname{cosec} 3\alpha)^2 - (\frac{1}{2})^2 = R_1B^2$.

The sine formula applied to the triangle R_1BC gives

$$\sin^2\phi = \sin\theta \,.\, \sin 3\alpha + \sin^2\theta \qquad \text{.....(1)}$$

From this we see that $\phi <$ or $> 2\alpha$ according as $\theta <$ or $> \alpha$; R_1 and S_1 are therefore on opposite sides of T, the limit point for which $\phi = 2\theta = 2\alpha$. Further, equation (1) can be written in the form

$$\sin^2\phi - \sin^2(\theta+\alpha) = \sin\alpha \,.\, (\sin\theta - \sin\alpha)(1 + 2\sin\alpha \,.\, \sin\theta) + 2\sin\theta \,.\, \sin\alpha \,.\, \cos\alpha \,.\, (\cos\alpha - \cos\theta) \quad \ldots\ldots(2)$$

from which we see that if $\theta < \alpha$, $\phi < \theta + \alpha$ and $2\alpha - \phi > \alpha - \theta$ and so, if R_1 is between T and B,

$$\widehat{R_1BT} > \widehat{S_1BT} \quad \ldots\ldots(3)$$

We can shew similarly that the same result holds if R_1 is between T and C. Remembering that R_1 and S_1 are on opposite sides of T, we deduce that $\widehat{R_2BT} < \frac{1}{2}\widehat{R_1BT}$ and so, that

$$\widehat{R_nBT} < (\tfrac{1}{2})^{n-1}\widehat{R_1BT} \quad \ldots\ldots(4)$$

Therefore R_n and similarly S_n both tend to T and the iteration converges.

The fact that R_{n+1} is on the same side of T as R_n enables an empirical adjustment to be made to the position of R_{n+1} to give more rapid convergence; this was noted in the description of the construction.

B. THWAITES

2667. The converse of the theorem of Pythagoras.

Mr. J. E. Blamey (Note 2337, May, 1953) and Prof. V. Thébault (Note 2372, December 1953) have given proofs of the converse theorem which are independent of the theorem. Prof. Thébault drew attention to the fact that his proof can be based entirely on the theorems of Book I of Euclid's *Elements*, for *certain* triangles. I may also refer to my own Note 2233.

Without questioning the precise standing, geometrical or logical, of the step from

$$a^2 = c^2 - b^2 = (c-b)(c+b) \text{ to } (c-b)/a = a/(c+b),$$

I wish to present a proof, independent of the theorem itself, and of any ratio and proportion relationships, and of any circle properties; one, moreover, which applies to *all* triangles ABC for which it can be asserted that $a^2 = c^2 - b^2$.

I assume the identity $c^2 - b^2 = (c-b)(c+b)$.

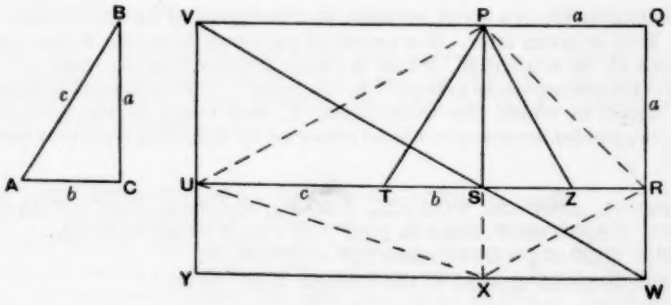


FIG. 1.

Construct a square $PQRS$ of side a. Produce RS to T and U so that $ST=b$ and $TU=c$. By the method suggested by Euclid I. 43, construct a rectangle $USXY$ equal in area to the square $PQRS$.

Then $$a^2=PQRS=USXY=(c+b)\,.\,SX,$$

whence $$SX=c-b.$$

Let Z be the point on UR, on the side of S remote from T, such that $SZ=SX=c-b$. Then $TZ=TS+SZ=c=UT$, and PT is a median of the triangle UPZ.

The fact that UP is parallel to XR follows from the equivalence of triangle $PUX(PUS+\frac{1}{2}USXY)$ and the triangle $PUR(PUS+\frac{1}{2}PQRS)$. Hence $\angle SRX=\angle PUS=\angle VPU$. Also from the congruent triangles SRX and SPZ, we have $\angle SRX=\angle SPZ$. Hence $\angle SPZ=\angle VPU$, and so, by adding $\angle UPS$ to each, we find that $\angle UPZ$ is a right angle.

Now in the right-angled triangle UPZ, PT is a median; hence, either by a "circle" or by a Book I argument, $PT=\frac{1}{2}UZ=c$. Thus the triangle PTS has sides equal to a, b and c; and so it is congruent to the triangle ABC. But the angle at S is a right angle, by construction. Hence the angle at C is a right angle.

It may be noted that the same construction, built up round a triangle PST in which the angle at S is known to be a right angle and the sides PS, ST, TP are of lengths a, b, c respectively, leads to the fact that $SZ=c-b$ and so $SX=c-b$, and hence $a^2=c^2-b^2$. A. G. SILLITTO

2668. Comments on "A note on infinite series" (Note 2568).

Note 2568 by Mr. F. H. Cody is concerned with the theorem that the condition $a_n\to 0$ is not sufficient to ensure the convergence of a series of positive terms Σa_n. A proof of the theorem is obtained by constructing a series of positive terms such that $a_n\to 0$ while

$$s_n=a_1+a_2+a_3+\ \dots\ +a_n\to\infty.$$

The theorem is evidently of considerable importance to the beginner (who is only too apt to run away with the idea that the theorem is false); and, for this reason, if for no other, it is desirable to use the simplest available divergent series which is capable of effecting the proof. Mr. Cody states that those text-books which mention the theorem explicitly usually appeal to the harmonic series $\Sigma(1/n)$ to effect the proof. It seems to me that the harmonic series is quite unsuited for the purpose because I consider that each of the various extant proofs of its divergence presents difficulties which appear formidable to a beginner.

As a simple example, Mr. Cody proposes the series in which

$$a_n=\log\frac{n+1}{n},\quad s_n=\log(n+1),$$

so that $$a_n\to 0,\quad s_n\to\infty.$$

This example has the undeniable merit of assigning simple values to a_n and to s_n; but I cannot help feeling a slight bias against using logarithms in preference to more elementary functions. Moreover, if I had to explain the example to a student, I should very soon find myself on the horns of a dilemma. The proof of the divergence of s_n to ∞ would not worry me unduly; but it would be quite otherwise with the discussion of the convergence of

a_n to 0, and this may account for the bias mentioned above. The simplicity of the value of a_n would inevitably lead the student to regard it as "obvious" that the limit of a_n is $\log 1 = 0$. Either one would agree with him and say nothing; thereby one would be deluding him with the impression that the *limit* of a function was *necessarily* equal to the *value* of the function at the limit, in symbols

$$\lim_{x \to c} f(x) = f(c).$$

There is no need for me to labour the undesirability of this procedure, because Hardy has already done it time and again. Or, if one preferred the other horn, one could attempt to supply the student with a more or less formal proof that the limit of a_n actually is log 1. At that stage of his development he would not appreciate the necessity of such a proof (in effect, that a logarithm is a continuous function of its variable) with the probable result that what he would regard as a quibble on my part would give him a permanent distaste for the subject.

It is fortunately possible to avoid this dilemma by choosing a different example and dealing with it appropriately; this example was given by A. de Morgan, *Differential and Integral Calculus* (1842), 235. The procedure used by de Morgan seems to have been overlooked subsequently, but it was recently rediscovered by Miss M. K. Hinds; see the *Gazette*, XXXII (1948), 212. Let k be any number in the range $0 < k < 1$, and take

$$a_n = n^{-k}$$

so that, for $n > 1$, we have

$$a_m > a_n$$

when $m < n$, and then

$$\begin{aligned} s_n &= 1^{-k} + 2^{-k} + 3^{-k} + \ \ldots \ + n^{-k} \\ &> n^{-k} + n^{-k} + n^{-k} + \ \ldots \ n^{-k} = n^{1-k} \to \infty, \end{aligned}$$

while $a_n \to 0$. This example, discussed by the method due to de Morgan constitutes what I regard as being obviously a "best possible" proof of the theorem. Of course the theorem that Σn^{-k} is divergent when $0 < k < 1$ is of venerable antiquity; but the proofs other than the proof given by de Morgan depend on comparison with the harmonic series or on more abstruse considerations, and so they cannot be regarded as appropriate for our purpose.

If the example with a general k is considered to be too abstract for a beginner, there is no need to do more than to take the special case $k = \frac{1}{2}$, and then only quadratic surds are involved. An example of the same character, but slightly less simple and slightly less attractive, is obtained by taking

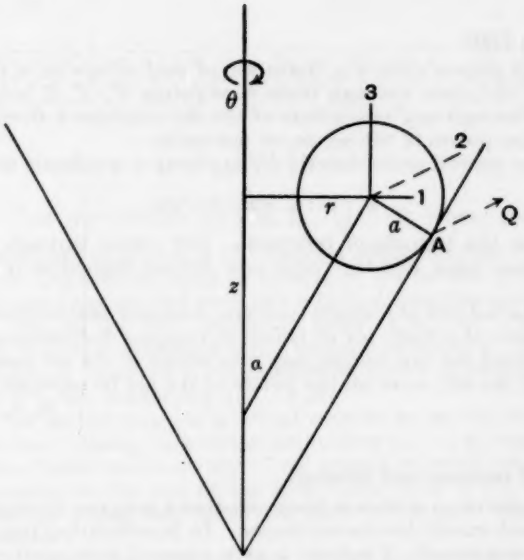
$$a_n = \frac{1}{\sqrt{(n+1)} + \sqrt{n}} = \sqrt{(n+1)} - \sqrt{n}, \quad s_n = \sqrt{(n+1)} - 1.$$

G. N. WATSON

Editorial note. The series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \ldots$. in which the sum to n terms exceeds N, and the nth term is less than $1/N$, if n exceeds $N(N+1)/2$, provides a simple proof without introducing surds.

2669. Sphere rolling on a cone.

A uniform sphere rolling without slipping on the inner surface of a circular cone whose axis is vertical has a constant vertical resolute of angular velocity. I have not seen explicit mention of this property in any text-book, and it does not appear to be capable of a simple proof comparable with that available in the case of the normal resolute of angular velocity where the sphere rolls on a spherical surface. I do not know of any proof simpler than the following.



Use cylindrical coordinates r, θ, z for the centre of the sphere, $r=0$ being the vertical axis of the cone, whose semi-vertical angle is α. Angular momentum equations are formed in terms of ω_1, ω_2, ω_3, the angular velocities of the sphere about axes of a right-handed frame rotating about the upward vertical with angular velocity $\dot\theta$, the axes being respectively in the directions δr, $\delta\theta$, δz. M is the mass of the sphere, a its radius, C its moment of inertia about any diameter, and Q is the resolute in the direction $\delta\theta$ of the reaction at the point of contact A. The equations quoted are of linear and angular momentum (about diameters) and equations expressing the fact that the point A of the sphere has zero velocity.

$$M(r\ddot\theta+2\dot r\dot\theta)=Q,\quad C\dot\omega_1-C\omega_2\dot\theta=aQ\sin\alpha,\quad C\dot\omega_3=aQ\cos\alpha,$$
$$a(\omega_1\sin\alpha+\omega_3\cos\alpha)=-r\dot\theta,\quad a\omega_2=\dot r\operatorname{cosec}\alpha.$$

The fourth equation is replaced by the result of differentiating with respect to the time: the equations are then multiplied respectively by $-a^2$, $a\sin\alpha$, $a\cos\alpha$, $-C$ and $C\dot\theta\sin\alpha$ and added. We get

$$(C+Ma^2)(r\ddot\theta+2\dot r\dot\theta)=0,$$

whence $Q=0$ and ω_3 is constant.

The same result holds when the sphere rolls in contact with the outer surface of the cone, the vertex being now uppermost. It is not necessary for the

sphere to be uniform, provided that the centre of mass is at the geometric centre and the moment of inertia is the same about every diameter. At first sight it might be thought that the result would be unaffected by rotation of the cone about its axis, either at constant prescribed angular velocity or with freedom for self-adjustment, but this is not found to be true owing to the necessity to introduce all three resolutes of angular velocity into the equations used. Of course steady motion is possible in which r, z and all angular velocities are constant.

H. W. UNTHANK

2670. On note 2486.

Mr. Collings proves that if a finite set of real points in a plane has the property that the conic through three base-points X, Y, Z (not members of the set) and through any two points of the set contains a third point of the set, then all the points of the set lie on one conic.

This may be proved more directly by applying a quadratic transformation

$$x' : y' : z' = yz : zx : xy,$$

where XYZ is the triangle of reference. For conics through X, Y, Z are transformed into lines, and the result now follows immediately from Gallai's theorem.

While on the subject of Gallai's theorem, does any reader know the answer to this problem—if a finite set of points in complex 3-dimensional space has the property that the line joining any two points of the set passes through a third point of the set, must all the points of the set be coplanar?

E. J. F. PRIMROSE

2671. Runs of successes and failures.

A psychologist often wishes to know whether a long run of correct responses by his rat could merely be due to chance. In investigating this problem, the result below was found. I submit it as a piece of pure mathematics; it is not well suited to the practical problem.

Suppose a penny is tossed $r+x$ times. How many runs of exactly r heads are to be expected?

If the run of exactly r heads occurs in the first r tossings, the $(r+1)$th tossing must give a tail. There are 2^{x-1} different ways in which the remaining tosses can result. The same number is obtained if the last r tosses yield the run. For intermediate positions, the run of heads must both be preceded and followed by a tail. There are 2^{x-2} ways for the remaining tosses.

Thus, if x is so small that it is impossible for two runs of r heads to occur, out of the 2^{r+x} possible outcomes, there will be $2 \cdot 2^{x-1} + (x-1) \cdot 2^{x-2}$ cases in which a run of exactly r heads occurs. Thus we have $(x+3) \cdot 2^{x-2}$ cases, provided x is at least 1. If $x=0$, there is of course just one way in which the run can occur.

But the above argument is still useful for larger x. If we imagine the 2^{r+x} possible cases written out, the argument tells us that there will be 2^{x-1} cases in which tosses 1 to r constitute a run, 2^{x-2} in which 2 to $(r+1)$ constitute a run, and so on. If a case contains two runs, it will appear twice in the above list; if three, three times. But this is just what should happen if we are attempting to determine the total number of runs in the 2^{r+x} possible cases.

Thus, for all x greater than 0, the total number of runs of precisely r heads in the 2^{r+x} possible cases for $(r+x)$ tossings is $(x+3) \cdot 2^{x-2}$. Beginning from $x=0$, the early numbers of the sequence thus defined are 1, 2, 5, 12, 28, 64.

Canterbury College, Christchurch, New Zealand. W. W. SAWYER

2672. On note 2449.

There is an extension of these two figures, the flexagon and the hexahexaflexagram, to a general figure using $3n$ colours. The flexagon is given by $n=1$ and the hexahexaflexagram by $n=2$. The general figure will consist of $9n$ triangles. It will be found that the strip has $\frac{3n}{2}$ twists. There are two cases to consider according to whether n is odd or even.

The case of n odd.

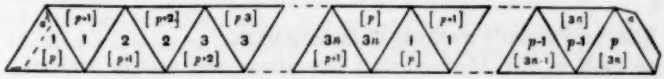


FIG. 1.

Colouring. Let the colours be 1, 2, 3, ... p ... $3n$ where $p=\frac{3n+1}{2}$. The strip is then coloured as shown in the diagram where numbers denote colours, the numbers in brackets referring to colours on the reverse of the strip. On the front starting from the end each two adjacent triangles are coloured with one of the colours in the following order 1, 2, ... p ... $3n$ continuing 1, 2, ... $(p-1)$ with the remaining single triangle coloured p. On the reverse the first triangle is coloured p, thereafter each two adjacent triangles are coloured $(p+1)$ ... $3n$, continuing 1, 2, ... $3n$.

Folding. The second triangle is folded over to lie on top of the first with the sides coloured 1 facing each other, and colour $(p+1)$ showing. The third triangle is then folded to lie on top of the second showing colour 2, and the process is repeated to the end of the strip. The ends of the strip are then joined together by gluing flap (a) to the space (a') so that the colours p are facing each other.

The resulting triangular flaps are then opened and flattened out to form a hexagon having 3 sets of $(n-1)$ hinged triangular flaps to each side. (When $n=1$, in the case of the flexagon, there are no flaps.) These flaps can be folded over to give n of the colours on either side of the hexagon, the remaining n colours being found similarly after turning the figure inside out as in the hexahexaflexagram.

This figure is a one-sided surface. Each colour on one side of the hexagon may be taken with any of $2n$ colours on the other side, and there are $3n^2$ combinations.

The case of n even.

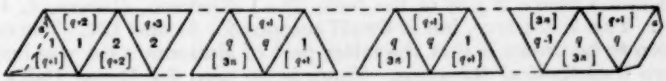


FIG. 2.

Colouring. Let the colours be 1, 2, 3, ... q ... $3n$ where $q=\frac{3n}{2}$. The strip is coloured as shown in the diagram. On the front starting from the end each two adjacent triangles are coloured with one of the colours in the following order 1, 2, ... q, repeating this sequence twice more to complete the strip.

On the reverse the first triangle is colour $(q+1)$ followed with each two adjacent triangles coloured $(q+2)$ ... $3n$ continuing with $(q+1)$, $(q+2)$ ... $3n$, repeating this latter sequence to the last single triangle which is coloured $(q+1)$.

Folding is done in the manner previously described.

This is a two-sided surface. The combinations of colours are $3n^2$ as before.

The figures for $n=3$, $n=4$, and $n=5$ have been made.

There is no limit to the value of n (other than the number of colours available).

Royal Holloway College, Englefield Green. JOAN CRAMPIN

2673. Note on Fermat's methods of factorisation.

Rouse Ball in his *Mathematical recreations and essays*, 2nd ed., (1892)*, states (p. 23) "It would seem that Fermat possessed some means of telling from its form whether a number was prime or not", and he gives two reasons for this belief, namely:

1. When Mersenne wrote to Fermat asking whether $n=10\,08955\,98169$ was a prime, Fermat replied that it was the product of 898423 and 112303.

2. Mersenne's list of the values of p for which 2^p-1 is prime, for $p\leqslant 257$. Rouse Ball knew only of one omission from Mersenne's list and says "it could not be obtained empirically, and it is impossible to suppose that it was worked out for every case".

As to (1), Rouse Ball was not aware of the context of Mersenne's question; Mersenne mentioned n as the last factor of an alleged pluperfect number, stated the other factors of the latter number, and asked Fermat what was the order of this pluperfect number, that is, the ratio of this number to the sum of its divisors. With this clue, it is easy to find the factor 898423 by a direct process. For details of it, see Dickson, *History of the Theory of Numbers, I*, 36 and footnote 317. There seems little doubt that that was Fermat's method.

As to (2), it is now known that for $61\leqslant p\leqslant 257$, Mersenne's list contains several errors; some that Mersenne said were prime are composite, and vice-versa. See Robinson, "Mersenne and Fermat numbers", *A.M.S. Proc.*, 5, 842 (1954).

It would seem that Mersenne and Fermat found by ordinary calculations which values of $p\leqslant 37$ make 2^p-1 a prime, using the theorem stated without proof by Fermat in 1640 that every prime factor of 2^p-1 is of the form $2kp+1$ (Dickson, *History, I*, 12) and probably also using Fermat's method of factorisation, discovered about 1643, which is mentioned below. Fermat wrote in 1640 that 47 divides $2^{23}-1$, and that 223 divides $2^{37}-1$.

Mersenne in 1647 stated without proof that 2^p-1 is a prime if p is one of the forms $2^{2m}+1$ and $2^{2m}+3$. If that were true (in fact it is not), it would account for most of the primes in Mersenne's list, namely, 5, 7, 17, 19, 67, 257. Mersenne continues with an obscure statement which seems to mean that 2^p-1 is also a prime if p is of the form 2^n-1 (Dickson, *History, I*, 13 and note 61). This is not true, but it would account for 31 and 127. The natural conclusion to be drawn is that the latter part of Mersenne's list was based on these two hypothetical theorems. There is no need for Rouse Ball's belief to account for Mersenne's list.

Finally, there are two grounds for rejecting Rouse Ball's belief.

(*a*) Fermat actually discovered a method of factorising numbers; this is to find by trial values of x and y satisfying

$$n=x^2-y^2.$$

* The two sentences quoted from the 2nd edition have been omitted from the 11th edition, (1939).

He took $x = |\sqrt{n}| + 1, +2, \ldots$ and found if $x^2 - n$ was a square, by extracting its square root. He shortened this process by considering whether the residue of $x^2 - n$ (mod 100) was consistent with its being a square. (Dickson, *History*, *I*, 357, quotes Fermat's description verbatim, as contained in a letter dated about 1643).

If Fermat or any of his friends afterwards discovered a better way of finding whether n is or is not a prime, it is probable that it would have been mentioned in some publication or letter.

(*b*) Fermat expressed his belief that every number $2^{2^n} + 1$ is a prime but admitted he had no proof (Dickson, *History*, *I*, 375). This is true for $n = 1$, 2, 3 and 4, and it may be presumed that Fermat knew this. As Euler later proved, it is not true for $n = 5$. If Fermat ever discovered a better way of finding whether n is a prime or not, surely he would have applied it to $2^{2^5} + 1$.

A. E. WESTERN

2674. The exponential series derived from the law of growth.

If a sum of money is growing continuously by the addition of simple interest at the rate of £r per £1 per annum, then if the amount after t years is £P, where P is a function of t, the interest on this amount from $t = 0$ to t is

$$\int_0^t Pr\,dt. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To find the total interest on £1 for t years at £r per £1 per annum compound interest, the interest being added continuously.

The simple interest on £1 for t years is

$$\int_0^t 1 \times r\,dt = rt.$$

This amount rt has itself been growing from $t = 0$ to t and the interest on it is

$$\int_0^t rt \times r\,dt = \frac{r^2t^2}{2}, \quad \text{using (1).}$$

Similarly, the interest on this $\frac{1}{2}r^2t^2$ from $t = 0$ to t is

$$\int_0^t \frac{r^2t^2}{2} \times r\,dt = \frac{r^3t^3}{3!},$$

and so on. Hence, adding in the original £1, we obtain for the amount £A after t years

$$A = 1 + rt + \frac{r^2t^2}{2!} + \frac{r^3t^3}{3!} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

in which each term after the first is the interest on the previous term. The relation to e^{rt} is completed in the usual way from

$$dA = A\,r\,dt,$$

whence

$$\log A = rt + c;$$

so that, if $A = 1$ when $t = 0$, $\log A = rt$ and $A = e^{rt}$.

W. J. FEARNS

2675. Notes on conics. No. 21. The oblique pedals of the focus.

In the paper on oblique pedals published in the *Gazette* Vol. XXXVIII (Sept. 1954), the principle that was being illustrated required the oblique focal pedal of a conic to be inferred from the familiar normal pedal, the auxiliary circle. It is the more necessary to insist that the oblique pedal is given directly by an argument which is perhaps the simplest of the many that have been used in the special case, no modification being required.

If Z is a point of intersection of two circles, and if three lines through Z cut one of the circles again in A_1, B_1, C_1 and the other again in A_2, B_2, C_2, the triangles $A_1B_1C_1$, $A_2B_2C_2$ are similar. In particular,

If two circles intersect in Z and A, and if two lines through Z cut one of the circles again in B_1, C_1 and the other again in B_2, C_2, the triangles AB_1C_1, AB_2C_2 are similar.

Now if the tangent p at a point P of the conic cuts the directrix s in Z, the four points Z, S, P, M are concyclic. It follows that if any circle through Z and S cuts p in Y_θ and s in X_θ, the triangle $SY_\theta X_\theta$ is similar to SPM, and therefore $|SY_\theta| = e\,|X_\theta Y_\theta|$. In other words, supposing X_θ to be fixed and P variable,

The point Y_θ on the variable tangent p such that the cross (SY_θ, p) is congruent with the fixed cross (SX_θ, s) is on the Apollonian locus $|SO| = e\,|X_\theta O|$.

Conversely, if Y_θ is any point on this locus, let the circle $SX_\theta Y_\theta$ cut s in Z, let the line through S perpendicular to SZ cut ZY_θ in P, and let PM be the perpendicular from P to s. Then the points Z, S, P, M are concyclic, and therefore the triangle SPM is similar to $SY_\theta X_\theta$, whence $|SP| = e\,|MP|$, implying that P is on the conic; by the construction, PZ is the tangent at P, and $(SY_\theta, ZY_\theta) \equiv (SX_\theta, ZX_\theta)$.

If Y_θ is on the Apollonian locus $|SO| = e\,|X_\theta O|$, there is one tangent p to the conic such that $(SY_\theta, p) \equiv (SX_\theta, s)$.

The identification of the oblique pedal with the Apollonian locus is complete.

The Apollonian locus $|SO| = e\,|X_\theta O|$ comes early into the investigation of the conic, since it determines the intersections of the conic with the line y_θ through X_θ parallel to the focal axis. If P is not on y_θ, then $|MP| < |X_\theta P|$ and therefore if P is on the conic, $|SP| < e\,|X_\theta P|$. It follows that the only points of the conic on the Apollonian locus are on the line y_θ, and that the points of the conic which are not on this line belong to the same one of the two regions into which the Apollonian locus divides the plane as the focus S.

Since the conic does not cross the Apollonian locus, the two must touch at any point which they have in common. This conclusion is easily verified. Every circle through S and X_θ is orthogonal to the Apollonian locus, and therefore if P is on y_θ, the line PZ, the tangent to the conic, since it is a diameter of the circle $SX_\theta P$, is a tangent to the Apollonian locus.

It is instructive to state results explicitly for the different types of conic. If $e > 1$, the Apollonian locus is a circle which has X_θ inside it and S outside. The line y_θ cuts the circle in two distinct points, whatever the position of X_θ on s, and the conic touches the circle at these two points; every other point of the conic is outside the circle. The hyperbola is the envelope of its oblique pedals.

If $e < 1$, the Apollonian locus is a circle with S inside and X_θ outside. If $|XX_\theta| < b$, that is, if the distance of the line y_θ from the focal axis is smaller than the minor semi-axis, y_θ cuts the circle and the circle has double contact with the conic; if $|XX_\theta| > b$, there is no contact. In any case the circle surrounds the ellipse. The ellipse is the envelope of those of its oblique pedals whose angles have cosines not greater than the eccentricity. For values of the cosine between e and 1, the oblique pedals compose two families of nested circles.

The oblique pedals of the focus of a parabola are the tangents themselves,

the tangent at Q being the locus of the point Y_θ on p such that $(SY_\theta, p) \equiv (SQ, q)$.

There is a moral in this note for the teacher of calculus : I fear that few of us would have a right to expect from our pupils an accurate answer to the simple problem,

Show that the equation

$$c^2 - 2cy - 3(x^2 + y^2 - a^2) = 0$$

represents a real circle for every value of c, and find the envelope of the family of circles given by this equation with c as a parameter. E. H. N.

2676. Can numerical integration be exact?

In many numerical procedures a function is specified only by a set of discrete ordinates for particular values of the argument.

It follows that strictly speaking the function is not then determined at all for other arguments, and that procedures like interpolation and numerical quadrature which appear to determine unique values for other arguments can do so only by in effect choosing a particular one out of all the functions which have the specified ordinates in common. The function so obtained may or may not be the function which the ordinates were intended to represent. In some cases it may not even approximately represent the intended function; this can happen even when the intended function and its derivatives are all continuous.

This is commonly appreciated in regard to interpolation with an insufficiently close set of ordinates. Analytic treatments of this question can be found, but they are difficult ; in practical work, an explicit or implicit check is ordinarily made that interpolates in sufficient agreement are deduced from different selections of the specified ordinates. This check can be inadequate if the intended function includes terms which have zero values at the specified ordinates, and non-zero values elsewhere. Ordinarily, there is no need to expect difficulties of this latter type.

One case where this latter effect does however arise is in the *Fourier analysis* of a discrete set of equidistant ordinates. This procedure is in fact more properly to be considered as trigonometric interpolation than as Fourier analysis in the full sense of the term, since only a finite number of harmonic terms can be determined, and these only with certain limitations.

If harmonics of sufficiently high order are involved, they will be indistinguishable from lower harmonics *in their effect at the arguments for which the equidistant ordinates are taken.*

In such cases, the remedy is to use a number of ordinates greater than the number of significant harmonic terms concerned, but even so it is necessary to have some ground for assurance that higher terms of indistinguishable type are in fact negligible. With $2N$ ordinates for $\theta = 0, \pi/N, 2\pi/N, \ldots$, we cannot detect the presence of a term in $\sin kN\theta$ ($k = 1, 2, 3, \ldots$) ; nor is there any way of deciding whether a set of $2N$ unit ordinates represents a function $\cos 2kN\theta$ ($k = 1, 2, 3, \ldots$) rather than a unit constant. In the latter case the unit constant gives a non-zero integral, but the functions $\cos 2kN\theta$ all give zero integrals, between limits corresponding to any pair of the ordinates concerned.

One of the most convenient formulae for numerical integration is the Gauss formula, or corrected trapezoidal rule, which expresses an integral in terms of equidistant ordinates throughout the range of integration, together with mean central differences of odd order at the terminal points. In the usual notation we have

$$I_n=\int_{x_0}^{x_n} f(x)dx=(\delta x)[(\tfrac{1}{2}f_0+f_1+f_2+\ \dots\ +f_{n-1}+\tfrac{1}{2}f_n)-\tfrac{1}{12}(\mu\delta f_n-\mu\delta f_0)$$
$$+\tfrac{11}{720}(\mu\delta^3 f_n-\mu\delta^3 f_0)-\tfrac{191}{60480}(\mu\delta^5 f_n-\mu\delta^5 f_0)+\ \dots\ ].$$

A derivation of the formula will be found in any standard text on numerical methods or finite differences.

When this formula is applied to a function $f(x)$ which has symmetry with respect to both the limits of integration (and is therefore periodic), as for instance $F(\sin^2 x)$ for $x_0=0$, $x_n=\frac{1}{2}\pi$, all *mean* central differences of *odd* order vanish at both limits, and the formula in this case reduces to the trapezoidal rule *without* difference corrections.

We might conclude that the trapezoidal rule is thus exact, and not an approximation only, for functions of this type. This conclusion is verified in many particular cases ; for instance, with $f(x)=\sin^2 x\cos^2 x$ and ordinates at $x_0=0$, $x_1=\pi/4$, $x_2=\pi/2$, we have, correctly, $I_2=\pi/16$. Many similar verifications can be found.

For $f(x)=\sin^4 x\cos^4 x$ with ordinates again at $x_0=0$, $x_1=\pi/4$, $x_2=\pi/2$, however, we have $I_2=\pi/64$, whereas the true value is $3\pi/256$. We may here write $f(x)$ in the equivalent form

$$(3-4\cos 4x+\cos 8x)/128\ ;$$

the two non-constant terms of this expression do not affect the integral, but they do affect the ordinates. For the $\cos 4x$ term, the ordinates are affected in such way that the Gauss formula is still correct, since alternate ordinates are affected equally and oppositely. For the $\cos 8x$ term, however, all the ordinates are equally increased, to give a corresponding error in the computed integral.

This question is not entirely academic. If the Gauss formula (or any equivalent formula) is applied to compute a complete elliptic integral, taking equidistant ordinates in a range of integration from $x=0$ to $x=\frac{1}{2}\pi$, the integrand is in fact of the form $F(\sin^2 x)$ and the difference-terms will in fact vanish as above discussed ; but the integrand will have an infinite series of Fourier components some of which will operate similarly to the $\cos 8x$ term in $\sin^4 x\cos^4 x$. The result obtained will then differ from the true value by the combined effect of all the terms of this type. The effect can (and should) be reduced by reducing the spacing of the ordinates.

This question could also be discussed in relation to the behaviour of the remainder term of the Gauss formula (this term must obviously tend to a finite non-zero limit when the formula is in error as discussed), or in terms of the Euler-MacLaurin expansion. Goodwin* has done this in relation to use of the Gauss formula for $\int_{-\infty}^{+\infty} f(x)e^{-x^2}dx$, with arbitrary $f(x)$, where a similar effect arises. A less sophisticated treatment is however possible, as above, in the analogous periodic case. THOS. H. O'BEIRNE

1876. THE MATHEMATICAL ASSOCIATION HAS FAILED.

Nevertheless the influence of Euclid's work has been tremendous ; probably no other document has had a greater influence on scientific thought. For example, modern high school geometries are usually modeled after Euclid's famous work (in England, Euclid is still used as a textbook).—Raymond L. Wilder, *The Foundations of Mathematics* (published in New York in 1952). [Per Dr. R. C. H. Young.]

* Goodwin, E. T., *Proc. Camb. Philos. Soc.* 45 (1949), 241.

REVIEWS

Algèbre. Tome I. Équivalences, Opérations, Groupes, Anneaux, Corps. (Cahier Scientifique No. 20). By P. DUBREIL. Pp. xii, 468. 3900 fr. Second edition, 1954. (Gauthier-Villars, Paris)

As the first edition of this excellent book was not reviewed in the *Gazette*, the second will be discussed here in its own right (with only incidental indications of how it differs from the first). The author's Introduction promises a second volume, " qui se rattachera étroitement au premier ", on groups with operators, linear algebras and Galois theory (besides, possibly, others on ideal theory and " hypercomplex numbers "); but regrettably, though Volume I was published in 1946, no successor has yet appeared, so the present review can be no more than an interim report.

Volume I is concerned with fundamentals : the emphasis is on sets, relations, correspondences, operations and " regular equivalences ", while the remaining topics listed in the chapter headings comprise groups, semigroups, fields, rings and algebraic equations. The first chapters include a very full treatment of the general theory of relations, special attention being given to partial and total orderings (and to their link with lattice theory). No previous knowledge is assumed, but the author expects a certain mathematical maturity in his reader, who is nowhere patronised, except perhaps by a completeness which leaves little to his imagination (even the slightest of theorems being accorded the dignity of a full-dress proof). At the beginning this might be welcomed by a reader not yet accustomed to abstract thought, but it is surely out of place in the later pages. And such a reader without previous algebraic experience is in any case hardly more than a polite fiction : though the book is, technically, self-contained, it is essentially a polished and unified (if somewhat discursive) treatment for those already familiar, or at least acquainted, with what it unifies. The author's extremely general and abstract approach marks him out as an Algebraists' Algebraist, and, in the reviewer's opinion, the real beginner should look elsewhere for his first, and inevitably partial, glimpses of the structures whose broader perspectives are so well exhibited here. Thus, while the meatier topics of course deserve all the space they get, a more condensed treatment of some of the comparatively trivial intervening material (e.g. the lengthy proofs that various systems of postulates are equivalent) would make for a more even passage for the reader. However, it is fair to mention that the first edition was both more elementary and more homogeneous : the present variations in level stem mainly from the interpolation of more recent (and in itself very worth-while) material, to the extent of some 150 pages. Much of this newer material has not appeared in book form before ; the highlights include thorough investigations into the possibility of embedding non-abelian semigroups in groups (the abelian case being relatively easy) and into the problem of factorisation in semigroups and integral domains (with an account of Krull's criteria for unique factorisation, besides the usual Noetherian decomposition theory).

But in any case the book as a whole commands respect as a sample of careful creative exposition in the highest French traditions (and, in this context, of course the second adjective weakens the first not at all). Any apparent faults are the natural concomitants of the book's chief virtues. If one regrets being slowed down by over-detailed trivialities, one has nevertheless to be thankful for the crystal-clear discussions of deeper questions ; similarly, a sustained freshness of approach more than compensates for a few slightly mystifying unconventionalities. One can often guess with tolerable accuracy what an author of lesser calibre will be saying for several pages ahead. But this book follows none of the well-worn grooves. The author has interesting points

to make even on the most standard topics, and the book is full of illuminating comments (usually as footnotes) on matters which sometimes cause confusion. There are also numerous exercises for the reader, of all grades of difficulty (and relevance); to give an example of the author's occasional mild eccentricities, one is, in a book avowedly on Algebra, perhaps entitled to raise an eyebrow at his exhaustive discussion of Pell's equation, crammed into one glorious footnote (to an exercise) occupying more than two pages. As Professor Julia says in his Preface, " le livre de M. Paul Dubreil diffère sensiblement des traités d'Algèbre déjà parus ".

A particularly valuable feature is the greatly expanded collection of five appendices or " Notes ". Note IV is a motivated resumé of the definitions of some of the most important kinds of spaces (euclidean, metric, complete, Hausdorff, topological and projective). The other four are in effect a 48-page sub-textbook on those parts, and just those parts, of Set Theory (in the broadest sense) needed to justify the " intuitive " ideas and arguments which every mathematician uses continually, while being only too often unconscious even of the possibility of putting them on a more satisfactory footing. Note I establishes the equivalence of the properties of ordinal and cardinal finiteness (as respectively defined by Denjoy and Tarski) with the property of being doubly well-ordered, and a correspondence is then set up in Note II between the " finite " sets (defined via any of these three methods) and the " natural numbers " as constructed from Peano's postulates. Note III deals with the Axiom of Choice and some of its equivalent forms (including the inevitable " lemma " of Zorn). These first three Notes are as stylish as anything in the book, and are the best *ab initio* treatment the reviewer has seen: they should be prescribed reading for all graduates about to begin research in Pure Mathematics. Quite apart from their vital role as foundation material, and from their intrinsic beauty as examples of the purest kind of mathematical reasoning (one may perhaps single out here the proof of the equivalence of the Axiom of Choice with the Well-ordering Principle), they set a standard of clarity which could hardly be surpassed.

There is a good index, and the typography conforms with the traditions of the publishers. However, the reviewer did notice twenty-four misprints (none particularly troublesome); and equation (3) on p. 340 is rather unfortunately expressed.

In short, after its 500% increase in price over the first, probably few private individuals will aspire to secure this second edition for their own shelves. But, as a brilliant and highly organised exposé of the fundamentals of algebra, with much to offer both to novice and expert, it should certainly be in every college library. Volume II will now be still more eagerly awaited.

M. P. DRAZIN

Grundzüge der Tensorrechnung in analytischer Darstellung. III Teil: Anwendungen in Physik und Technik. By A. DUSCHEK and A. HOCHRAINER. Pp. vi, 250. 41s. 1955. (Springer, Vienna)

This is the final part of the authors' work, the second part of which was published in 1950 (*Gazette*, XXXV, 1951, p. 69). The present small volume does not set out to be a comprehensive text-book of applied mathematics and mathematical physics, but seeks to develop such basic pieces of theory as are suitable for treatment by tensor methods. These form in themselves quite an extensive field of study.

The book makes an agreeable impression. While it lacks the leisured, lengthy elegance of a Salmon's *Conics* or of a Goursat's *Cours d'analyse*, it likewise avoids the staccato style favoured by some present-day mathematical

writers, and is written in a manner that is simultaneously clear, concise and unhurried. Moreover, the authors are in no way obsessed by fascinations of notation, part of their stated purpose being to persuade the reader that tensor methods do not apply to *all* physical problems : they show no disposition to use tensorial sledgehammers for the purpose of cracking infinitesimal nuts. For the most part they use cartesian systems, but make free use whenever necessary of the full tensor analysis developed in the earlier volumes. The text is illustrated by diagrams.

The present volume covers an impressive range of topics, including the dynamics of a particle and of rigid systems, elasticity, hydromechanics, heat conduction, electromagnetism and relativity. A special feature of the book, to which the authors call attention in the preface, is the development of a theory of " double vector fields " (*vektorielle Doppelfelder*). Such a field is defined over a given spatial region by two vector fields

$$A_i(x_1, x_2, x_3), \quad B_i(x_1, x_2, x_3),$$

together with a non-vanishing scalar field $\lambda(x_1, x_2, x_3)$ such that $A_i = \lambda B_i$, the functions A_i, B_i, λ being all of differentiability class C^2. This general theory has a number of particular applications of which only one example need be mentioned here, namely the electrostatic double field having as basis the vectors of electric field-strength and electric displacement.

The book ends with four sections on relativity, the special theory occupying thirty pages and the general theory twenty-seven. In this small compass the authors contrive to give an account of the Lorentz transformation both in its three-plus-one-dimensional physical aspect and in its four-dimensional geometrical aspect, without confining themselves to relative motion parallel to a common x-direction ; and also an account of special-relativity mechanics and electrodynamics. In the general-relativity section they deal with the law of gravitation, with the Schwarzschild solution, with planetary motion, the motion of perihelion and the bending of light-rays, and, finally, with the de Sitter and Einstein worlds.

Books of this kind have something in common with mince pies. Their contents are rich and finely divided, but have common qualities making for a unity of flavour. Also they occupy only a small volume. The extent to which such richness is digestible must depend upon the capacity of the consumer. That several such books (in various languages) have appeared in recent years is almost certainly a sign that an interest in tensor methods is growing among technologists. In this country, indeed, the formation some years ago of the Tensor Club of Great Britain gives clear proof of the existence of such an interest. Books like the one under review, especially when written by people of the eminence and experience of Professors Duschek and Hochrainer, are therefore likely to meet a real need, and are calculated to inspire practical physicists and technologists to pursue further the application of tensor methods to their own specialist branches. The present book does seem to be an admirable one of its kind, and is to be warmly recommended.

H. S. RUSE

The slide rule for students of science and engineering. By T. G. C. WARD and G. W. BLAKEY. Pp. 94. 3s. 6d. 1955. (English Universities Press)

An explanation of the use of the slide rule for beginners in Science and Engineering which caters for the trigonometrical needs of the sixth form. The authors state that the book has been tested at Sandhurst and at Welbeck College.

R. L. G.

Advanced calculus. By L. BRAND. Pp. xii, 574. 68s. 1955. (John Wiley, New York ; Chapman and Hall)

This is a " second course " in calculus, suitable for degree students, dealing with differential calculus of functions of one or more variables, vectors, integration, including line and multiple integrals, and two concluding chapters on complex variable and on Fourier series. Professor Brand does not state explicitly that the book is for applied mathematicians, but the contents and the omissions seem to suggest this intention. Thus the only integral mentioned is Riemann's; the chapter on number fields is short (25 pp.) and sketchy ; the complex function $u+iv$ is studied only under the assumption that the four partial derivatives u_x, ... are continuous, so that Cauchy's theorem is proved by an application of the two-dimensional Green's theorem and topological considerations are somewhat crude ; vectors are given a geometrical emphasis. Granted the intention, this is perhaps all to the good, for it enables the author to discuss in careful detail topics, of considerable importance for the rigorous treatment of applications, which are often very cursorily treated in the " Calculus for engineers and physicists " type of book. Thus the existence of implicit functions is dealt with, first for $F(x, y)=0$, then for $F(x, y, z)=0$, then for two functions in four variables, and finally for three functions in six variables ; the conditions under which two successive limiting processes may be reversed in order are carefully discussed in a chapter on uniform convergence, introduced not for its own sake but because such reversal is a valuable tool which we must learn to handle with care as well as with skill, and examples of its power are exhibited. The reader is not likely to be left with the feeling that differentiation under the integral sign is so subtle and delicate a process that it should be attempted only as a last resource in examination anaesthesia. Fourier series are confined to the comparatively simple and useful case of piece-wise continuous functions.

These remarks should make plain the opinion that this book is not suitable for the embryo professional analyst ; indeed the author implicitly admits this by taking as his sub-title " An introduction to *classical* analysis ". But the high-class technician should find it exceedingly useful ; it gives him just what he wants to know, with just the right amount of rigour to enable him detect fallacies, his own or those proffered to him from the practical side, without losing too much self-confidence.

Mathematical printing in the U.S.A. is now much improved from that of the pre-war years ; but even so, the craftsmanship of our University Presses has not been equalled, and in this beautifully produced but expensive book there are lapses from the best standards. The diagrams are on rather too small a scale, and sufficient emphasis is not laid on the essentials.

T. A. A. B.

Foundations of Quantum Theory. By ALFRED LANDÉ. Pp. v, 106. $4.00. 1955. (New Haven : Yale University Press, London : Oxford University Press)

This extremely interesting and expensive little book by Professor Landé has as its purpose the construction of a systematic derivation of the general principles of quantum theory from a set of postulates with direct physical significance and of universally admitted validity. The reviewer was particularly interested in this book since he himself attempted to solve the same problem over twenty years ago. Professor Landé has independently adopted a very similar approach to the problem but has strongly reinforced the arguments advanced by the reviewer by reference to the principles of thermo-

dynamics and the separation of a mixture of gases by means of a semi-permeable membrane. This line of argument, like that advanced by the reviewer, indicates, even if it does not definitely establish, the possibility of representing quantum states by projective operators in a Hilbert space. The difficulty is to show that the transition probabilities are related to the scalar products of the unit vectors representing different quantum states. Here Professor Landé gives the most interesting thermodynamic argument which certainly makes the known formula of the quantum theory more plausible. The third chapter of Professor Landé's book leads up to the exchange relations which are based upon new definition of conjugate operators. Although the book is not intended as an introduction to beginners, but as a meditation on the subject for those who already have some familiarity with the technique, I think it will be welcomed by many students of the subject as throwing a fresh light on the abstract principles of the quantum theory. G. TEMPLE

Einführung in die Geometrie der Waben. By WILHELM BLASCHKE. Pp. 108. Swiss fr. 15.25. 1955. (Birkhauser, Basle and Stuttgart)

The theory of honeycombs (or webs, as they are often called) is a comparatively new branch of geometry, so it may be worthwhile to give a brief idea of what a honeycomb is. The simplest type, a 3-honeycomb in a region of a plane, consists of 3 families of plane curves with the property that exactly one curve of each family passes through a given point and that, in the region considered, two curves of different families have exactly one point of intersection. The simplest example consists of 3 pencils of lines, the region being in this case the whole projective plane.

The development of the theory is mainly due to Professor Blaschke and his fellow-workers during the 1930's. This small book gives a very clear account of the main ideas of the subject, and after reading it one wonders how the author has managed to deal with so much in about 100 pages. One of the fascinations of the subject is that it links up with so many other branches of mathematics, and the author has given several examples of this in the course of the book.

This is the fourth volume in the series *Elemente der Mathematik vom Höheren Standpunkt aus*. This series is meant for the non-specialist, so that the aim is to assume as little previous knowledge as possible : the first volume, which was also geometrical, satisfied this condition perfectly. This book, on the other hand, is likely to prove difficult for the non-specialist, as it assumes a fairly extensive knowledge of differential geometry.

The printing is excellent, and there are over 40 beautifully drawn figures.

E. J. F. PRIMROSE

Théorie Mathématique du Bridge à la Portée de Tous. By ÉMILE BOREL and ANDRÉ CHÉRON. Pp. xvii, 424. Second edition. 2,200 fr. 1955. (Gauthier-Villars, Paris)

This solemn treatise on the calculation of probabilities in the game of bridge is written jointly by an eminent professor of probability theory and a well-known bridge player and journalist. It reminds us that probability theory was founded three centuries ago in the correspondence of Pascal and Fermat about a problem submitted to them by the Chevalier de Méré whose slight miscalculation of the chances in a dice-throwing game had lost him his fortune at the gaming tables. Using a modern mechanical computing machine (based on the principle first applied by Pascal) the authors have computed the probabilities of every possible card distribution that might arise at any stage in the playing of a hand of bridge. All their results have

been calculated anew ; any results already published have been used only as checks. Many of these earlier results were found to be approximations based on the use of logarithms or on Stirling's approximations for factorial numbers. But in this treatise all results have been computed arithmetically and have been verified by computing them again by alternative methods. In most of the tables probabilities are given as percentages correct to 6 decimal places.

The book was originally published in 1940 ; in this second edition a few misprints have been corrected and a " note " of 30 pages on practical application of Bayes's Theorem has been added.

Although the book is addressed to " everyman " the authors assume that the reader has a knowledge of elementary probability theory. They first analyse every phase of the game in five descriptive chapters (250 pages)—On Shuffling the Cards, The Distribution of Cards after the Deal, The Bidding, The Play, Rules of Scoring (Auction and Contract) and the Best Line of Play. Then, for more mathematical readers, there are ten " notes " on particular problems and generalisations (150 pages). It is emphasised that the book is not a treatise on how to play bridge but is purely an analysis of the probabilities that arise as they appear to the three players (excluding " dummy ") each of whom has a different problem ; for example, no note is taken of the information conveyed by bids or by bidding conventions. The authors would claim only that the 131 tables which summarise the results of their analyses give added precision to the bridge player's judgements when, for example, he considers how to play " le squouize " or whether to bid a " grand chelem ". But the treatise has also the more serious purpose of demonstrating methods of computing probabilities, of generalising some of the methods used, of exposing some common fallacies and, not least, of proving that " the realm of psychology, in the shades of which players believe they find refuge from the mathematician and his calculations and the threat with which he menaces their liberty, cannot escape his sovereign grasp and that the formula of Bayes takes into account the most subtle and insidious tactics of their play ".

This is the most comprehensive and authoritative treatise on the mathematical aspects of bridge ever likely to be published. Professor Borel seems to have extracted from the game everything that could be of interest to the " probabilist " and M. Cheron has tabulated everything that the bridge player could possibly want. This book could be regarded as marking the end of an epoch in the development of the Theory of Games—here, in effect, is the complete solution of the problem put to Pascal by the Chevalier de Méré. The modern theory of probability and its applications arose almost as a by-product during the solving of an apparently trivial question. The new epoch has already opened with a problem posed a few years ago at the poker tables of Princeton—a problem which has already led to the founding of a more generalised mathematical Theory of Games and Strategy and to the promise of equally important by-products. Happily, Émile Borel has contributed to that also.

B. C. BROOKES

Differential Equations. By LESTER R. FORD. New 2nd Edition. Pp. xii, 291. 37s. 6d. 1955. (McGraw-Hill)

It is unusual to find Clairaut's equation and simple examples of solution in series in the first chapter of a text-book on differential equations, but the idea is a good one. From the beginning the student learns that successive approximation to a solution may be the best we can do and that singular solutions may exist.

Subsequent chapters cover special methods for equations of first order, linear equations of any order with a brief account of the use of the Laplace

transform, solution in series of the hypergeometric, Legendre's and Bessel's equations, approximate numerical solutions, and two chapters on partial differential equations. Numerical solutions are preceded by a good chapter on finite differences, including approximate differentiation and integration and the algebra of operators Δ and E. General solutions of simple types of partial differential equations are obtained before separation of variables is used to solve problems of vibration and the Laplace equation in two dimensions. Linear dependence of functions receives considerable attention. It is perhaps a pity that the author does not illustrate the statement "that the vanishing of the Wronskian is not alone sufficient has been proved by an example for the case $n=2$" on p. 87; at first sight this appears at variance with the theorem on p. 63.

The book may be recommended as a good blend of the scholarly and the practical approaches to its subject. The student reading without the help of a tutor will not find it easy. Although answers are given to all exercises, a larger proportion of worked examples in the text would be an advantage. The only misprint noted is an obvious u for n on p. 120, line 2.

C. G. PARADINE

The Mathematical Mind. By M. L. CARTWRIGHT. The James Bryce Memorial Lecture. Pp. 27. 2s. 6d. 1955. (Oxford University Press)

Next to creating mathematics or rediscovering the creations of others nothing is more fascinating than hearing great mathematicians talk about their mathematical life, its joys and sorrows. Miss Cartwright not only shows us something of the way in which her own discoveries were made but tells us something also of the experience of other great minds.

The lecture, to a mixed audience of mathematicians and laymen, begins by drawing some distinctions between mathematics and research in fields outside the sciences; mathematicians are judged by their papers not their books; mathematics is the subject in which originality is more sharply defined.

Is mathematics a field of discovery or invention? Miss Cartwright comes down with Hadamard on the side of discovery, but it is hard to believe that the concepts of mathematics, like *bounded variation* or *measurability* existed before they were invented any more than did the alphabet. To the less gifted how much consolation is there to be found in Miss Cartwright's confession that reading and understanding mathematics is slow and difficult even for the expert? The late Paul Dienes once complained to Borel that he had to read mathematics with a pencil in his hand and found, instead of sympathy, surprise that he had not realised that all mathematicians work in the same patient manner.

What is good mathematics? Miss Cartwright finds it in a result which can be stated boldly and simply without many qualifications, a result which is not easy to prove but the repercussions of which are immense; in proofs and results which have an element of unexpectedness and in those which seem inevitable once stated; in the long complicated chain of reasoning leading to a desired end as well as in the technique which achieves its end with the minimum of effort.

Amongst many other stimulating observations Miss Cartwright reminds us that inspiration may be false as well as true, that conviction is needed to achieve discoveries and that even becoming aware of imperfections is progress in research. Good ideas fall like rain on many heads but only few have done the necessary preparatory work to use the idea and make mathematics out of a momentary impression.

R. L. GOODSTEIN

linear equations of any order with a brief account of the use of the Laplace

Functional analysis. By F. RIESZ and B. Sz.-NAGY. Translated from the 2nd French edition by L. F. Boron. Pp. xii, 467. $10. (65s.) 1955. (Ungar Publishing Co., New York, Blackie & Son Ltd., Glasgow)

Leçons d'analyse fonctionnelle appeared first in 1952, with second and third editions in 1953 and 1955; this rapid production endorses the opinion of experts that the work is outstanding among treatises on linear analysis. It is in two parts: the first, by Riesz, deals with real variable and integration, the Lebesgue integral being defined by Riesz's own process; the second part, by Nagy, starts with integral equations and proceeds to discuss the main problems of linear spaces and operators. For further details, we may refer to the review of the first edition in the *Gazette*, XXXVII, pp. 157–8. It is difficult to believe that any competent reader would find trouble with the limpid clarity of the French original; but for those who need a translation, this appears to be accurate and lucid, and the volume has been beautifully printed and produced (though I think there should be a reference to Nikolsky on p. 217). T. A. A. B.

L'Enseignement des Mathématiques. By J. PIAGET, E. W. BETH, J. DIEUDONNÉ, A. LICHNEROWICZ, G. CHOQUET and C. GATTEGNO. Pp. 173. 12 Swiss fr. 1955. (Delachaux and Niestlé, Neuchatel and Paris)

This important collection of essays is the first combined publication by the international commission for the improvement of the teaching of mathematics and is therefore of special interest to members of the Mathematical Association.

The main standpoints represented in these essays are those of modern algebra, symbolic logic and operational psychologism. Beth poses the dilemma of the teaching of modern mathematics when he reminds us that school studies are almost exclusively concerned with quantification logic (first order predicate calculus) in which the concept of set plays no part, and yet the difference between the system of rationals and the system of real numbers cannot be described in quantification logic. Dieudonné considers abstraction in mathematics and follows the development of the number concept from signed integer to the axioms of a commutative field. Lichnerowicz, in a plea for the introduction of the spirit of modern algebra into elementary geometry, stresses the need to discard outmoded concepts and theories to make room for new ideas in the school course and instances the hodograph as a notion which became dispensable with the differentiation of vectors. Choquet seeks to base school geometry on a system of axioms which are not simply logical but are also reasonably close to the experience of each of us. A relation of congruence is taken as a primitive notion, and the class of all point couples congruent to a given pair of points is called the length of the member couples of the class. The familiar properties of a congruence relation are supplied by axioms and an axiom assures us that for any congruent pair AB, $A'B'$ there is an isometry (one-one transformation of the plane into itself such that if $X \to X'$, $Y \to Y'$ then XY and $X'Y'$ are congruent) which transforms A into A' and B into B'. Perpendicularity is defined in terms of symmetry and angle in terms of the common part of concentric "semicircles". Gattegno's contribution discusses teaching patterns for presenting mathematical ideas to different age groups designed not only to teach technique but also to *evoke* the desired concept in the pupil, without which the subject must remain a meaningless gibberish. Gattegno attaches great importance to the teaching of the *reversibility* of the operations of elementary algebra.

The first essay in the collection, and the only one written with a knowledge of the other contributions is by Jean Piaget on "Les structures mathématiques

a momentary impression. R. L. GOODSTEIN

et les opératoires de l'intelligence", in which he expounds his now well known view that the fundamental concepts like number and group are "primitive" intuitions present already in the child's mind. Piaget does not make the mistake of taking the laws of logic as the laws of thought (i.e. a property of brains) but he considers that certain laws of logic have an impulsive force for the men of certain civilisations. The edifice of mathematics rests on "structures" which *correspond* to actual structures in the mind, the operations of mathematics deriving from *actions* which are linked in structural form in the mind. Opposing the standpoint of naive psychologism, Beth remarks that (with certain reservations) we can say, with Frege, that psychology has scarcely made any contribution to the solution of the problems in the foundations of mathematics. R. L. GOODSTEIN

Proceedings of the Second Symposium in Linear Programming. Washington, D.C. Jan. 27-29, 1955.

The first Symposium on Linear Programming, i.e. the determination of the maximum of a linear form, subject to inequalities in the variables of the form, was held in June 1949 in Chicago and led to the publication of the Cowles Commission Monograph No. 13 entitled "Activity Analysis of Production and Allocation". That book contained the first printed reports on the Simplex Method, which is the most powerful routine hitherto devised for solving the problem, on some special cases of Linear Programming such as the "transportation problem", and on the relations between Linear Programming and Theory of Games, in addition to papers on convex sets and on topics of interest to economists.

The present two volumes, issued in mimeographed form by the National Bureau of Standards and the Directorate of Management Analysis of the U.S. Air Force gives valuable information on how the subject has developed since then. (It is also gratifying to find that the clumsy title of the earlier publication has not been universally accepted as an alternative to the short if inaccurate term "Linear Programming").

The proceedings were divided into five parts, as follows: Applications (12 papers), Economic Theory (6), Computation (8), Theory of Linear Inequalities (6), Developments in Linear Programming (one, by G. B. Dantzig, the discoverer of the Simplex Method). As always in such conferences, the allocation of some papers is arbitrary, although in this case the referees of the papers have done a very efficient job. There are two excellent talks introducing Parts I and III by W. W. Jacobs (Military Applications of Linear Programming) and by A. J. Hoffman (How to solve a Linear Programming Problem) respectively. Dantzig's concluding talk contained much of interest to the expert, as did many other papers.

It is pleasant to watch a young and vigorous subject in its steady progress that does not yet show any sign of fatigue. S. VAJDA

Arithmetic for Engineers. By C. B. CLAPHAM. 5th revised edition. Pp. xiii, 540. 21s. 1955. (Chapman and Hall)

The scope of this book, first published in 1916, is somewhat wider than the title suggests, since it included from the beginning some algebra, and appendices on (i) trigonometry of right-angled triangles, (ii) verniers and micrometers, were added later. This new edition has an "addendum" comprising some notes on the original text, a collection of harder problems and the solution of pairs of simple simultaneous equations. The earlier algebra covered simple indices, negative numbers and graphs, simple equations and transposition of formulae.

of the other contributions is by Jean Piaget on " Les structures mathématiques

The book is written for " the elementary student of average ability and for those engaged in home study, who need a patient and full explanation of every step . . . so that they may advance surely, though slowly." The author succeeds very well in his purpose. Short lessons are followed by plentiful examples of machine-shop mathematics, the data very often expressed in engineers' terms and drawings. There is no doubt that the practical nature of the problems set would be an incentive to the type of student the author has in mind and there are probably many boys at school who would respond to this important, though elementary, kind of applied mathematics.

The only misprint noted is the sign + for × on p. 98, l. 11 from foot. The degree sign is sometimes omitted from an angle, as in tan 13 and $\sin(90 - A)$. Conversions to percentages sometimes lead to such erroneous statements as $(1{\cdot}05 \times 100)/5 = 21\%$. The terms trapezium and trapezoid are used for what we now call quadrilateral and trapezium respectively.

C. G. PARADINE

Statistics. By L. H. C. TIPPETT. 2nd edition. Pp. vi, 224. 7s. 6d. 1956. (Oxford University Press. London, Cumberlege)

Many teachers will have read the first edition of this volume in the Home University Library series and remember it as a sensible and informative account of the basic ideas of statistics and their range of application. The new edition uses as illustrations figures published in the nineteen-fifties and mentions such activities as operational research and linear programming. The author is careful not to over-state the case for statistics, confessing to some disillusionment as to its effectiveness, when applied to social conditions and economics, in revealing laws which could justly be described as scientific. Nevertheless in his closing sentences he looks forward " to the day when statistics will occupy a place in education only a little way behind arithmetic . . . and such general introductory books as this will become obsolete." If this day should dawn the subject will probably rank as mathematics because of the arithmetic involved. The restricted amount of statistics which has already found its way into the syllabuses of examinations in mathematics is regarded with distaste by many teachers and pupils. One reason for this is perhaps the necessity for accepting formulae without proof. The mensuration of the sphere could be cited as an example where this is the usual procedure, but the notion of a surface area or a volume is more obvious than that of a standard error or correlation coefficient. This book, used as supplementary reading to text-book or lesson, should help many to acquire the outlook of the statistician.

C. G. PARADINE

High Speed Aerodynamics and Jet Propulsion. Vol. I. Thermodynamics and Physics of Matter. Editor F. D. Rossini. 812 pp. £5. 1955. (Oxford University Press)

This is the first volume of a series entitled " High Speed Aerodynamics and Jet Propulsion ". The remaining volumes are devoted to more technological subjects. The present volume contains ten independent monographs. Some are new, others are mere transcriptions of previous publications.

A. Fundamentals of Thermodynamics (108 pp.) is a useful exposition at undergraduate level. C. Thermodynamic Properties of Real Gases and Mixtures of Real Gases (99 pp.) consists mostly of rather useless and very uninteresting empirical formulae without an idea. Work of the past fifteen years is almost completely ignored. Of 75 references 60 are American. The remaining sections are authoritative monographs on special fields mostly of the high standard associated with the Handbuch der Physik or Reviews of Modern Physics. They include B. Fundamental Physics of Gases (80 pp.), E. Critical Phenomena (82 pp.), F. Properties of Liquids and Liquid Solutions (53 pp.),

G. Properties of Solids and Solid Solutions (92 pp.), H. Relaxation Phenomena in Gases (90 pp.), I. Gases at Low Densities (43 pp.), J. Thermodynamics of Irreversible Processes (20 pp.).

From this list of topics it is clear that the volume covers a wide and interesting field, but not so wide as to justify the title. The words "Physics of Matter" mean the whole of physics except radiation and particles. In fact little, if any, space is devoted to the electric or magnetic or optical properties of matter. Not even the mechanical and thermal properties are covered. For example neither classical wave motion nor turbulent motion is mentioned; nor is thermal expansion. "Thermodynamics and Kinetic Theory of Matter" would be a less misleading title.

I cannot help wondering why these ten independent monographs are bound into a single volume. I would far rather buy separately the ones that interest me at say £1 each than pay £5 for the lot. If it is designed to compel customers to buy more than they want, I doubt whether in the long run this kind of racket will pay. If on the other hand there is the faintest suggestion that such a volume contains all, or nearly all, the physics required by research workers employed on jet propulsion, such a suggestion would be vigorously contested by every eminent physicist in the world. If ever such a suggestion gains acceptance in any country, it will be a gloomy outlook for research in that country.

The printing is up to the recognized Oxford standard. No more need be said.

E. A. GUGGENHEIM

The Convolution Transform. By HIRSCHMANN & WIDDER. Pp. x + 268. 45s. 1955. (Princeton University Press. London: Cumberlege)

This book makes a very welcome and timely appearance. In it the two leading workers in the field of convolution transforms set out for the general mathematical reader the main lines of the theory and its principal developments.

The convolution transform is the name given to the transform

$$f(x)=\int_{-\infty}^{\infty} G(x-t)\,\phi(t)dt$$

which generalizes the "faltung" process. Formal arguments show that if the function $E(s)$ is connected with the kernel $G(t)$ by

$$1/E(s)=\int_{-\infty}^{\infty} e^{-st}G(t)dt,$$

then $E(D)f(x)=\phi(x)$, where $D\equiv d/dx$, and an appropriate meaning is given to the operator $E(D)$. This relation between the inversion function $E(s)$ and the kernel $G(t)$ has led to the theory developing in terms of $E(s)$ rather than of $G(t)$; and the necessity of representing $E(D)f(x)$ in terms of elementary operations involving D has led to the choice of

$$E(s)=\exp(-cs^2+bs)\prod_{1}^{\infty}(1-s/a_k)(\exp(s/a_k),\quad a_k, b, c \text{ real}, c\geqslant 0,$$

since by the Laguerre-Polya theorem, this function is the uniform limit of a sequence of functions each of which has but a finite number of zeros.

The scope of the book may be briefly indicated from the contents of the chapters. In Chapter I, the principal ideas are illustrated by means of examples including the classical ones of the Stieltjes and Laplace transforms. In Chapter II, finite kernels, i.e. kernels $G(t)$ for which the corresponding

function $E(s)$ is a polynomial, are treated, and their connection shown with the Green's function of a finite differential system. In Chapter III, non-finite kernels, with $c=0$, are discussed, and the elegant Obrechkoff proof of the Laguerre-Polya theorem is given.

Chapters IV and V are devoted to a study of the kernel $G(t)$. The authors give an intrinsic description of it by connecting it on the one hand with Schoenberg's work on variation-diminishing and totally positive functions, on the other by showing its relation to certain elements of statistical theory. Asymptotic properties of $G(t)$ and its derivatives are obtained when (1) all the zeros of $E(s)$ lie in a half-plane and (2) the zeros of $E(s)$ do not lie in a half-plane. These cases are intrinsically different, and indeed lead to the classification of convolution transforms by the authors into essentially different types.

Chapter VI is devoted to real inversion theory, theorems being proved for each of the three types of convolution transform distinguished by the authors. An inversion theorem is called real if $E(D)f(x)=\lim_{n\to\infty} E_n(D)f(x)$ and $E_n(D)$ involves only the processes of differentiation or a shift of the variable, so that $f(x)$ need be known only for real values of the argument. Chapter VII deals with representation theorems, where necessary and sufficient conditions are all given in terms of the elementary operations $E_n(D)f(x)$.

In Chapter VIII the authors treat the Weierstrass transform where $E(s)$ is the special function $\exp(-s^2)$. The inversion operator is most appropriately represented by means of a contour integral. Further inversion theorems of the same type are discussed for more general $E(s)$ in Chapter IX. In both chapters, inversion and representation theorems are proved, and there are interesting references in Chapter VIII to the connection between the Weierstrass transform and the heat equation, for a full discussion of which however, the reader will have to consult Widder's paper (Trans. Amer. Math. Soc., Vol. 71). The concluding chapter is devoted to certain miscellaneous topics.

Very few errors have been found. There is a reference on p. 77 to a paper by G. Doetsch, which is not listed in the bibliography ; and an obvious error in para. (4.2) on p. 257.

The book is recommended as giving in readable form the essence of discoveries made during the past quarter century, and hitherto available only in journals. It contains a judicious choice from this material, and may be read by anyone whose mathematical equipment includes a good knowledge of real and complex variable theory. D. B. SUMNER

Trigonometrical Series. By A. ZYGMUND. Pp. 329. $1.50. 1955. (Dover Publications, New York)

A photostatic republication of this famous standard work, originally published in Poland, in 1935. A few of the rather numerous misprints have been removed. The reprint is cheaply priced and printed on reasonably good paper. The reviewer understands that a completely new and enlarged edition is being prepared by the author. W. W. ROGOSINSKI

Lectures on functions of a complex variable. By W. KAPLAN. Pp. 435. 1956. 80s. (Michigan University Press. London, Cumberlege)

This is a collection of 31 lectures given by various authorities in the field at a semi-international conference on functions of a complex variable held at the University of Michigan, Ann Arbor, in June 1953. Many of these lectures are of an expository kind surveying a wide range of topics. Nearly half of them present the fashionable abstract approach of topological, functional, or algebraic nature. In particular, Topology as basic background to complex analysis and with special reference to Riemann surfaces is well represented.

Of the more classical topics we may mention : conformal mapping, normal families, polynomial approximation, potential theory, and functions of several variables. It is, of course, quite impracticable to resurvey this survey in a review. Nor, in fact, would it be easy to find a competent reviewer to do this. Modern Analysis has extended so much in range and scope that it is now exceedingly difficult for any single analyst to comprehend the whole. Such conferences as that at Ann Arbor serve the important purpose of keeping up the strained communications between the many experts approaching a common field of research from different directions. To judge from these well presented lectures, the conference must have been a great success, and any serious analyst can now, through this welcome publication, join in its benefit.

W. W. ROGOSINSKI

Approximations for Digital Computers. By CECIL HASTINGS, JR. 32s. 1955. (Princeton University Press. London, Cumberlege)

This book deals with the subject of best approximation in the sense of Chebyshev (namely, least maximum deviation) to functions of a single real variable by means of polynomials and rational functions. The first part of the book is an account (presented in the form of a film strip with a running commentary under each frame) of an iterative method of adjusting the parameters in the chosen form of approximation (e.g., the coefficients in a polynomial) to arrive at the best fit—in which the maximum absolute deviation is least. The theoretical basis of the method is rather sketchily presented here but can be found (in the case of the polynomial form, for example) in " Leçons Sur L'Approximation des Fonctions d'une Variable Réelle " by De la Vallée Poussin (Paris, 1919). There is a section dealing with the choice of form of the approximating function where, for instance, matching the asymptotic behaviour of the function may need to be considered. This section ought, logically, to have come at the beginning. The second part of the book contains some seventy-odd approximations to the elementary and other useful functions, each presented with a carefully drawn error curve. While most of these are very suggestive many programmers for digital computers would have wished for more accuracy if this were possible (most of the approximations yield between 4 and 10 decimal places). However the book certainly earns a place in any numerical analysis library.

R. A. BROOKER

Darstellungen von Gruppen mit Berucksichtigung der Bedurfnisse der modernen Physik. By H. Boerner. Pp. xi, 287. Dm. 36.60. 1955. Grundlehren der Math. Wiss. (Springer, Berlin)

This book is concerned with those parts of the theory of group representations which have applications to Physics, but the treatment is mathematical throughout, no account being given of the physical applications themselves. It has in fact the same avowed purpose as Murnaghan's well known book on group representations, but the treatment here is quite fresh and the present book, which is considerably shorter than Murnaghan's, does not duplicate it in any substantial sense.

It is well written and great care has been taken in the ordering and presentation of the material. The reader will find in each chapter a chain of theorems, many of which will need careful study. He will also find however that the clear enunciation of the theorems, the way in which the chapters are broken up into comparatively short sections, and a valuable introductory summary to each chapter all help to make his reading easier.

The book begins with a chapter on matrices, giving an account of orthogonal, unitary and Hermitian matrices, transformation into diagonal form, direct sum and Kronecker product of two matrices, and irreducible systems of matrices. This is followed by a chapter giving a concise account of the sym-

metric and alternating groups and those continuous groups which enter into the physical applications. There are also sections on the exponential of a matrix, the infinitesimal ring of a linear group and integration in Lie groups.

Chapter III deals with representation theory in general. It begins with the reducibility of representations of a finite group and goes on to the irreducible representations and the characters of a finite group. The later sections are concerned with the representations and characters of continuous groups and there is a concluding section on groups which have a normal subgroup of index two. There follows a short chapter on the representations of the symmetric group. It deals with Young's standard tableaux and in particular with the calculation of the matrices of what is usually known as the natural representation.

Chapter V is concerned with the representations of the full linear, unimodular and unitary groups. It deals with the symmetry classes and the decomposition of tensor space and with the irreducible representations of the full linear group associated with standard Young tableaux. The chapter concludes with a full treatment of the rational, semi-rational and rational integral representations of the various continuous groups under consideration.

The next chapter returns to the characters of the symmetric groups and their relation to the characters of the full linear group. A method is given for the calculation of the characters of a symmetric group and there is a rather detailed account of the characters of the alternating group. In the account of the analysis of the Kronecker product of two irreducible representations of the full linear group it might perhaps have been appropriate to mention the lattice permutation rule for the multiplication of two Schur-functions.

Then follows a long chapter on the rotation group. The treatment here is novel as far as the textbooks are concerned, being based on papers of E. Stiefel published in 1942 and 1945. This involves the p-dimensional toroid associated with a maximal Abelian subgroup of the rotation group. The group of automorphisms of this toroid is related to a group generated by reflections across certain hyperplanes in p-dimensional Euclidean space. The required properties of the rotation group are then derivable from the group generated by the reflections. The diagrams of the hyperplanes are called Stiefel diagrams and are frequently employed throughout the later chapters. The book concludes with chapters on the spin representations of the rotation group and on the Lorentz group.

The author does not intend his bibliography to be complete, but it contains well over a hundred references. There is a good index. The book is well produced and although some typographical errors were noticed, none appeared to be of much consequence. The work is a valuable addition to the literature of group representation theory, more particularly in relation to continuous groups. There is of course no discussion of modular representations.

H. O. FOULKES

Algebraic Threefolds. By L. ROTH. Pp. 142. DM. 19.80. 1955. (Springer, Berlin)

This monograph in the *Ergebnisse der Mathematik* series provides a fully documented and up-to-date account of the present state of knowledge of the geometry of algebraic threefolds. All workers in this field will owe a great debt of gratitude to the author for a gallant attempt to give coherence to a theory which is still admittedly very incomplete and which abounds with unsolved problems. The author writes from the classical algebro-geometric point of view of the Italian school, but recognises that, at present at least, classical methods require the help of transcendental and topological theory at several points, and he gives a brief account of this theory wherever necessary. Chapter I deals with the invariantive theory, that is with the canonical system,

the arithmetic genus, the irregularities and similar topics, which can mostly be dealt with by methods which are a natural extension of those applicable to algebraic surfaces. Chapter II is a study of the theory of systems of equivalence, as developed by Severi, Todd, B. Segre and others. Chapter III deals with linear systems of surfaces on an algebraic threefold and in particular with the Riemann-Roch theorem on the freedom of the complete linear series characterised by a given non-singular surface on a non-singular threefold. The full title of the monograph is *Algebraic threefolds with special regard to problems of rationality*, and in the last three chapters the author deals at considerable length with the topics governed by the subtitle, to the study of which he has himself made significant contributions. An algebraic d-dimensional variety is said to be *unirational* if it can be mapped on an involution of order $n \geqslant 1$ in a linear space S_d, and it is *rational* in the case $n = 1$. For $d = 1, 2$ it has been shown by Lüroth and Castelnuovo that a unirational variety is necessarily rational, but for $d \geqslant 3$ this is no longer the case. The author gives in Chapter IV and elsewhere many criteria for unirationality and rationality, but a complete set of necessary and sufficient conditions remains to be found. It can for example be shown without difficulty that the non-singular cubic primal in S_4 is unirational, but whether it is rational or not is still an unsolved problem. Enriques and Castelnuovo have shown that any algebraic surface, on which the process of successive adjunction applied to any curve system on the surface terminates, is rational or scrollar. For a unirational variety V_d ($d \geqslant 3$) it can be shown that the adjunction process terminates, but the author produces counter-examples to show that there exist threefolds on which the adjunction process terminates but which are not unirational. Chapter V is mainly devoted to a study of the adjunction problem and of the interesting class of Fano threefolds W^{2p-2} for which the generic curve section is canonical of genus p. Chapter VI deals with threefolds which admit finite continuous groups of automorphisms and is based on the general theory of Lie. Although the book is primarily concerned with threefolds, many of the results which the author quotes or establishes also apply to varieties of higher dimensions, and he makes it clear whenever this is the case. Finally in a brief but useful appendix he gives a survey of the chief concepts and theorems underlying the work.

D. W. BABBAGE

Theorie der analytischen Funktionen einer komplexen Veränderlichen. By H. Behnke and F. Sommer. Pp. x, 582. DM. 69.60. 1955. Die Grundlehren der mathematischen Wissenschaften, 77. (Springer, Berlin)

Many books in many languages describe the classical theory of functions of a complex variable, the great Cauchy-Riemann-Weierstrass triptych ; nor, in spite of change of emphasis and importance, does it seem likely that this theory will in our age lose its high value for the professional mathematician whatever his speciality ; even if this were to happen, surely the beauty and power of the theory would save it from the dust of the museum.

Professor Behnke and Dr. Sommer have treated the three main streams fairly ; for instance, after Cauchy's theorem has been proved for functions which have a derivative and the topological troubles have been sorted out by proving the theorem first for a triangle, there is a long section on the calculus of residues and its applications. But on the whole, the emphasis is on the development of Riemann's ideas, and in many ways the new book is a supplement to, rather than a replacement of, the earlier book on function theory in this series by Hurwitz, completed by Courant. Thus the first half of the book, though not narrowly restricted to preparing solely for the developments in the second half, does nevertheless work steadily towards the three hundred pages in which conformal representation and Riemann's mapping theorem, the Riemann surface and the uniformisation theorem, functions and integrals

on a Riemann surface and Abel's theorem receive full attention. Fortunately for the novice, the authors take great pains to make the introduction of new ideas as simple and as concrete as possible, with plenty of illustrative material. Thus, though their formal setting-up of the Riemann surface is axiomatic, a Hausdorff space axiomatically defined being specialised first to a two-dimensional manifold and then further to a Riemann surface, this is not done until a number of simple Riemann surfaces have been considered more or less intuitively in some detail. The young student perplexed by the laconic brilliance of Weyl's *Idée der Riemannschen Fläche* would find his path illuminated by the less dazzling light supplied by this book.

Naturally this is no encyclopedia. There is no theory of special functions; lines of development suggested by, for instance, Picard's theorem or Nevanlinna's work cannot be followed up. But these things can be found elsewhere, and as it is, normal families of functions, harmonic functions, series of orthogonal functions, Hilbert spaces have all been pressed into service. Moreover, the study of Riemann surfaces and of functions thereon will bring the reader within striking distance of modern work on algebraic functions and integrals and their geometrical applications.

As always, it is a pleasure to commend the clear and faultless Springer typography. T. A. A. BROADBENT

Einführung in die Zahlentheorie. By A. Scholz/B. Schoeneberg. 2nd Ed. Pp. 128. DM. 2.40. 1955. Sammlung Göschen, 1131. (W. De Gruyter, Berlin)

This is a welcome new edition of A. Scholz's tract first published in 1945 and now revised by B. Schoeneberg. In this new edition the sections on quadratic residues and on quadratic forms have been slightly extended, while the chapter on computational techniques in the original edition has now been omitted.

The outstanding feature of this book is the treatment of quadratic residues and of quadratic forms with integral coefficients, which follows on the customary introduction into the divisibility and congruence properties of integers. Here within relatively little space a great deal of ground is covered. Thus two proofs of the law of quadratic reciprocity are presented, and classes of quadratic forms, representation of integers by such forms, reduction Theory, automorphic substitutions are discussed.

This little book should prove to be an attractive and stimulating introduction to Number Theory. It cannot attempt to touch upon all the main branches of the subject, but instead concentrates on a few principal topics and thus succeeds in leading the reader up to interesting fundamental results of a comparatively advanced nature. A. FRÖHLICH

Analytische Fortsetzung. By Ludwig Bieberbach. Pp. 168. DM. 24.80. 1955. Ergebnisse der Mathematik und Ihrer Grenzgebiete. (Springer, Berlin)

A power series $\sum_0^\infty a_n z^n$ with a finite radius of convergence R must have at least one singularity on $|z|=R$. A self evident corollary of this classical theorem is that a series such as $a_0+a_1z^p+a_2z^{2p}+a_3z^{3p}+\ldots$ must have at least p singularities, and cannot be regular on any arc exceeding the pth part of its circle of convergence. One might hazard amongst others the guess that a series $\sum_0^\infty a_n z^{\lambda_n}$ could not be regular on any arc of its circle of convergence if λ_n tends to infinity faster than n. Again the coefficients of the power series associated with the elementary functions such as $-\log(1-z)=\sum_0^\infty z^n/n$,

$(1+z)^\lambda = \sum_0^\infty \binom{\lambda}{n} z^n$, are represented by relatively simple formulae. It might then be expected that the question of analytic continuation of the function $\sum_0^\infty a_n z^n$ is connected in some way with the existence of a formula $a_n = A(n)$ in which the function $A(z)$ has some special characteristics. This is the type of question with which Bierberbach's book deals—a rather special part of the theory referred to in the title. Problems of this kind have been studied only for some 60 years and in fact are often referred to by personal reference to J. Hadamard. In this time they have been subjected to a many-sided investigation by mathematicians all over the world. In the bibliography are listed more than 300 papers by more than 100 authors. The main achievement of this book is the way in which the work of so many writers is developed systematically and with remarkable brevity. This is done by developing the subject as a particular exploitation of the Laplace Transform. Many special devices however are also needed and although complete proofs of most results are included, the first aim has been to include an account of all the work listed in the bibliography. A careful appraisal of the contributions of different workers leads to many interesting remarks of a personal, historical and philosophical kind.

The book forms an excellent introduction to the subject, as well as being an effective continuation of the author's encyclopaedia article and text books.

A. J. MACINTYRE

Applied Group-Theoretic and Matrix Methods. By BRYAN HIGMAN. Pp. 454. 60s. 1955. (Oxford University Press)

In 443 pages the author of this book attempts to initiate the beginner into finite groups, matrix algebra, hypercomplex numbers, group representation, numerical evaluation of determinants, operator space, the external forms and the internal structure of crystals, vibrations of molecules, factor analysis, correlation and error, continuous groups, eigen-value theory in Hilbert space, symmetric functions, Schur functions, tensors, relativity including the basic principles of general relativity, quantum theory, n-electron systems, molecular structure and spectra, and finally Eddington's quantum relativity. The Preface tells us that the book arose out of a course of lectures delivered to honours students in Physics and Chemistry in the University College of the Gold Coast and that the book is designed primarily for physical scientists and is intended to cover systematically the various fields of application of group theory and matrix theory.

The Preface also tells us that "the attitude to mathematical rigour has been that of the scientist. It is desirable, but an understanding of what is going on is more important". The author hopes the mathematician will try to get inside the head of the scientist before criticizing.

The task which the author has set himself is virtually an impossible one. To cover such an extensive programme in such limited space involves condensation so ruthless that the residue is not easily comprehended by the reader who is not already familiar with the subject matter. It is indeed very doubtful whether the novice will be able to make anything at all of the greater part of this book. The present reviewer being ignorant of crystallography eventually gave up the attempt to understand those chapters of the book devoted to this topic. To illustrate the speed at which the argument proceeds, the opening chapter of 9 pages after giving a number of realisations of finite groups devotes exactly two pages to theory in which we proceed from the definition of a group to Lagrange's theorem and to the theorem that every group of order n is a subgroup of the symmetric group of order $n!$ In proving that cosets are uniquely determined by a subgroup it is assumed but not explicitly stated that

$g \in G \Rightarrow gG = G$. This important fact however has not been previously referred to. Remembering that the book is written for scientists and not for mathematicians it might be permissible to state a theorem without proof, but of what value can a "proof" be either to a scientist or to a mathematician, which assumes that the reader is familiar with important results which are not even mentioned?

To take another example, we are introduced to integral transforms as follows (p. 192) in a section entitled "Differential operators and Hilbert space"—

"As a vector becomes a function defined along a line, so a matrix becomes a function of two variables defined on a plane, and the equation $Y = AX$ takes the form

$$g(y) = \int_{-\infty}^{+\infty} a(x,y)\, f(x)dx."$$

..."A function derived from another in this way is known as an integral transform."

A page later we read "this equation $\left[L\frac{di}{dt} + Ri + \frac{1}{c}\int_{-\infty}^{t} i\,dt = e(t)\right]$ is solved by multiplying through by e^{-pt} and integrating from 0 to ∞". One might ask how the beginner is expected to know that the integration is with respect to p, a symbol which has not previously appeared in this section. Can any one learn from the above what an integral transform is unless he is already familiar with the concept?

Not only are many of the arguments and even definitions very obscure but some are also quite misleading to the scientist (false to the mathematician). On p. 23 we read that "in two dimensions the most general real, orthogonal matrix has the familiar form $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \sin\theta \end{bmatrix}$. The reader must wait till p. 90 before he is disillusioned on this point. On p. 25 we are told that "there are p^n pth roots of an $n \times n$ matrix". One has sympathy for the scientist who decides to try this out on the matrices $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Again on p. 204 it is stated that "all matrices with the same eigen-values are transformable into one another" although matrices which are not reducible to diagonal form have been introduced on p. 72. The effect on the reader who discovers blunders such as the above is that he is reluctant to accept the validity of any statement without giving it the closest scrutiny. On the other hand the terse nature of the mathematical arguments renders such a scrutiny both difficult and tedious.

The task of digesting this book is not made any easier by the notation and terminology employed. The author invents a new notation to deal with row and column vectors. A column vector is denoted either by $X\dagger$ or by X^*. Its transpose and its conjugate transpose are then denoted by $\dagger X$ and $^*X$ respectively. The defect of this notation is that it does not carry over to matrices. Thus the transpose of a matrix A is denoted by $\widetilde{A}$, its conjugate by $\overline{A}$, and its conjugate transpose by $\widetilde{\overline{A}}$ or by $\overline{\widetilde{A}}$. To a reader accustomed to other notations this new notation is bound to be unnecessarily confusing. Again the author's use of the term "induced representation" pp. 155 and 182, to mean a representation by induced matrices rather than one induced by a representation of a subgroup, is at variance with current terminology. Similarly, the statement on p. 187 in the chapter on continuous groups that "infinitesimal operators always commute" sounds strange until one realises that the author is referring to infinitesimal transformations.

It seems to the reviewer that $+\sin\theta$ and $-\sin\theta$ in formula 12.3.1. should be

interchanged assuming that all the rotations mentioned have the same sense.

The printing is in accordance with the usual excellent standards of the Oxford University Press, but one is puzzled by the variety of founts used for definitions. One finds " group " in small type italics, " improper subgroup " in roman capitals, " characteristic equation " in ordinary italics, " proper expansion " in inverted commas and " conjugate element " in ordinary roman type. In spite of its many blemishes the experienced mathematician who is able to read between the lines will find a great deal of interest in this book. The author is clearly an expert at handling group representations, and the diversity of applications and illustrations which he provides is stimulating and illuminating. The last sentence of the Preface suggests that the book was rushed into print and its shortcomings mentioned above certainly suggest that this was so. One gets the impression that the author is very knowledgeable in a more than usually wide range of subjects, that he has terrific enthusiasm, but no patience. If he were to rewrite this book in five years' time, he might produce a really outstanding volume. D. E. RUTHERFORD

L'Existence en Mathématiques. By E. W. Beth. Collection de logique mathématique. Pp. 60. 900 fr. 1956. (Gauthier-Villars, Paris)

This volume brings together a number of lectures given by the author at the Sorbonne in 1954. The first chapter briefly reviews prior order predicate logic and the notion of a deductive theory. This is followed by a discussion of the Lowenheim-Skolem theorem from the standpoint of valuation theory; amongst the many very interesting applications of the theorem which Beth introduces are the striking results that first order predicate logic cannot express a necessary and sufficient condition for a class to be infinite and that Peano-Arithmetic cannot be based on a finite number of axioms (as was first shown by Ryll-Nardzewski in 1953). The principal new result is the theorem of partial formulae (a form of which was given by Gentzen in 1934) and this result is used to furnish a very simple proof of Herbrand's Theorem. In Chapter V there is an account of Brouwer's fan theorem ; in connection with a proof of this theorem Beth makes the important observation that Brouwer's proof of the fan theorem appears to be irreconcilable with his rejection of formalism. The book concludes with a discussion of Gödel's proof of the consistency of the continuum hypothesis and some ideas of Tarski's on the elimination of the axiom of infinity.

This is a fresh and invigorating set of lectures and though the reasoning is tight and the pace warm the author's gift of exposition easily carries the task through successfully. R. L. GOODSTEIN

Calculus : A modern approach. By KARL MENGER. Pp. xvii + 354. 3rd Ed. 1955. (Ginn & Co.)

This book presents a development of the ideas introduced in 1944 by the author in his *Algebra of Analysis* (reviewed in XXIX, p. 238). An earlier edition of Menger's Calculus was reviewed by Dr. Busbridge in XXXVIII, p. 76.

Menger's treatment is characterised by a complete revision of the traditional uses of variables, a revision which he has recently described in his article in the *Gazette* called " What is x and y ".

That there are serious confusions in the traditional uses of variables is beyond doubt, and much credit is due to Dr. Menger for his criticism ; but it may well be asked whether an elementary text book on the Calculus is the best place in which to feature these criticisms and the notational innovations to which they have led the author. R. L. GOODSTEIN

Initiation au Calcul Matriciel. By A. MONJALLON. Pp. 131. Fr. 700. 1955. (Vuibert, Paris)

This little book does what it sets out to do very well indeed. It provides the beginner, the first year University student, with a clear and interesting introduction to matrix theory ; the pace is slow, numerical illustration is plentiful and at the cost of a number of results assumed without proof, a sufficiency of application is achieved to convince the student of the value and importance of matrix algebra.

Although " Calcul Matriciel " is intended primarily for Science and Engineering students, it will make a useful preliminary course also for the Honours Specialist, for apart from a little want of elegance at one or two points (for instance in the theory of determinants) the book maintains a high standard of presentation.

R. L. GOODSTEIN

Euclid's Elements. Translated by Sir T. L. HEATH. 3 volumes. $5.85. 1956. (Dover Publications, New York)

This reprint is an unabridged and unaltered republication of the revised second edition (1925) with introduction and commentary by the Translator. It has been published through special arrangement with the Cambridge University Press. The Association for the Improvement of Geometrical Teaching, out of which the Mathematical Association developed, critized and opposed the use of the Elements as a School text, but no one doubts its value and importance to the mathematician, both for is great virtues and for its blemishes.

R. L. GOODSTEIN

Famous problems of elementary geometry. By FELIX KLEIN. Pp. 92. $1.00 paper bound, $1.50 cloth. 1956. (Dover Publications, New York)

This reprint is remarkably good value for three half-crowns. The famous problems discussed are the duplication of the cube, the trisection of an angle, the division of the circle into equal parts and the construction of the regular polygon of 17 sides, the transcendence of e and π. R. C. Archibald's notes in the second edition are included in the reprint ; they contain not only important corrections to small details in the text but numerous references.

R. L. GOODSTEIN

THE MATHEMATICAL ASSOCIATION

Intending members of the Mathematical Association are requested to communicate with one of the Secretaries, Mr. F. W. KELLAWAY, Miss W. COOKE. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of *The Mathematical Gazette* and a copy of each new report as it is issued.

Change of Address should be notified to the Membership Secretary, Mr. M. A. PORTER. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

Subscriptions should be paid to the Hon. Treasurer of the Mathematical Association.

The Library of the Association is housed in the Library of University College Leicester.

The address of the Association and of the Hon. Treasurer and Secretaries is **Gordon House, 29 Gordon Square, London, W.C.1.**

REPORT OF THE COUNCIL FOR THE YEAR 1956

Membership.

During the year ended 31st October, 1956, 145 ordinary members, 37 junior members and 2 life members were admitted to the Association. At the end of the year the membership figures were: Honorary, 5: Ordinary, 2,424: Junior, 105: Life, 244: a total of 2,778 compared with 2,699 at the beginning of the year.

It is with regret that the Council reports the death of the following members: Mr. C. J. Cozens (1923), Mr. J. P. Gabbatt (1910), Dr. Christine M. Hamill (1952), Mr. W. K. Macdonald (1926), Dr. F. G. Maunsell (1938), Mr. D. K. Picken (1905), Mr. A. Robson (1912), who was President in 1949, Mr. G. E. Strawson (1928); and also of Professor Emile Borel (who was an Honorary Member), and of another Honorary Member, Sir Edmund T. Whittaker, who was President 1919–1920.

During the year a printed Membership List has been published.

Finance.

Income this year was £4,431 17s. 10d. and expenditure was £5,081 5s. 8d., so that the General Fund was reduced by £649 7s. 10d. and now stands at £1,444 11s. 0d.

Some reduction in the General Fund was anticipated because of the production of the *Primary Schools Report* (£786 1s. 8d.), but the position was unexpectedly weakened when the Inland Revenue authorities suspended repayment of income tax under the covenant scheme. Over £400 of estimated income has therefore not yet been received. It is to be hoped that the repayment will eventually be made when the position, following a decision against the National Book League in a test case, has been analysed by the Inland Revenue authorities.

Sales of Reports reached over £1,000, and they promise to be even higher in the next account. The *Primary Schools Report* was reprinted just before the account closed. The Hon. Auditor has decided that the cost of reprinting (£463 10s. 0d.) should not be shown as an item of current expenditure, and the amount has been placed under Assets in the balance sheet. The cost of reprinting will, of course, be more than recovered by sales of the *Primary Schools Report* during the next accounting year.

The unexpected shortcoming of over £400 and the unexpected bill for £463 would have caused a heavy overdraft but for the generosity of Messrs. G. Bell & Sons, who made an advance payment to the Association of over £950; this payment, for sales of *Gazettes* and Reports, was not due until January, 1957.

Production of the *List of Members* cost a little over £300. Less than one half of the members who ordered a copy of the List had purchased one at the close of the account, and only £105 had been received from sales.

The death of Mr. A. Robson having left the Association with only one Trustee, Professor Jeffery, it was decided to increase the number of Trustees to three. Mr. K. S. Snell and Mr. J. T. Combridge have consented to act.

The Mathematical Gazette.

The number of notes submitted for publication is considerably in excess of the space available but there is a shortage of material in certain special categories, such as descriptive articles on current research.

The rising costs of printing the *Gazette* have obliged the Editor to discontinue the provision of bound offprints *gratis*, but offprints may be purchased at cost price.

The distribution of the *Gazette* has increased by some ten per cent during the past year.

Library.

The Library has received a quite substantial number of accessions during the year by the generosity of members and the relatives of past members; there have been no purchases of books or periodicals. Binding of periodicals is making steady progress. There has been an increase in the number of volumes borrowed, particularly by members of the Association living in the Leicester area. The collection has proved to be of great interest to Leicester students, many of whom have become members of the local branch of the Association.

The Teaching Committee.

Mr. J. T. Combridge has resigned from his position as Chairman of the Teaching Committee. His period of Office has been one of considerable activity in the sub-committees and this has owed much to the warmth of his support. But while he encouraged all possible speed, his tactful insistence on high standards of general production prevented any premature plunge into print. The Teaching Committee has placed on record its keen appreciation of Mr. Combridge's distinguished service. Mr. A. W. Riley has been appointed Chairman.

Sub-committees have continued their work on Sixth Form Algebra, on Sixth Form Analysis, and on the work in Modern Schools. It is anticipated that the report on Sixth Form Algebra will be published shortly, and the Modern School report is nearing completion.

A reprint of the *Report on the Teaching of Algebra in Schools* is now in the hands of the printer. A reprint of the *Report on the Teaching of Mathematics in Primary Schools*, called for within eight months of the original publication, was placed on sale in October. The heavy demand for this Report amply justifies the Association's extension of its work into this new field.

Plans are in hand with a view to the Association's participation in the International Congress of Mathematicians in Edinburgh in the summer of 1958.

The Committee has suffered a severe loss by the death of Mr. Alan Robson, who had taken a leading part in its activities during the past thirty years and was Secretary from 1927 to 1937 and Chairman from 1938 to 1946. Always zealous for mathematical integrity, Mr. Robson was a major contributor to almost all the Association's reports from the *Arithmetic Report* onwards; although since his retirement to Grasmere his appearances at meetings had been less frequent than formerly, his penetrating criticisms and deep wisdom were still freely available to sub-committees.

The Branches.

The Branches Committee met twice during the 1955–56 session and much useful information was exchanged between Branch representatives and secretaries. It is quite plain from these meetings and from the Branch reports which appear in the *Gazette* that the work of the Branches is flourishing. Programmes of talks and discussions are most varied, catering for a wide range of mathematical interests.

One feature of the work of the Branches is the increasing practice of arranging meetings for junior members and for members of the sixth forms of Grammar Schools. It is hoped that this early contact between scholars and the Association will lead to an increased membership of the Branches and, later, of the main Association.

The work of the Committee was, until August, most ably coordinated by Dr. Maunsell, Secretary of the Southampton Branch. Dr. Maunsell's death was a sad loss to the Association and will be felt particularly in the Branches Committee.

Problem Bureau.

There has been a slight increase in the number of applications received, some of them from members in distant parts of the world. Together with the usual inquiries for solutions to Cambridge and other scholarship papers, there have been others, many of them very interesting. The Bureau can still handle more work, and no member should feel diffident about using its services. Enquiries should be sent to Dr. G. A. Garreau, 90 Wyatt Park Road, London, S.W. 2.

Officers and Council.

Council wishes to offer, on behalf of all members, its sincere thanks to the President, Professor G. Temple, and to the Officers for the work which they have done for the Association during the year. Mr. J. T. Combridge will be proposed at the Annual General Meeting for the office of Vice President and thus resume his membership of Council.

CONFERENCE FOR TEACHERS OF MATHEMATICS IN SCHOOLS

held at Pembroke College, Cambridge, 30th July to 2nd August, 1956

The Faculty of Mathematics at Cambridge University and the Board of Extra-Mural Studies, jointly responsible for this Course, are to be congratulated on a highly successful venture. Forty-five mathematical teachers, for the most part Heads of Department and responsible for Sixth-Form work, attended by invitation and were nobly entertained. The intellectual banquet consisted of three main courses. The first was a series of lectures designed to refurbish and modernise the picture of University Mathematics which most of us had dimly stored in the recesses of our memory. Dr. Wylie compressed a masterly account of developments in Topology into three lectures; Dr. Cassels took an interesting journey through several fields of Number Theory in two; Dr. Taunt met us very nearly on our ground with two lectures on Vector Methods. This course was certainly stimulating, and, considering the depth of penetration into the subjects involved, the lectures were agreeably connected and compact. It must be confessed, however, that many found that a full hour slightly exceeded their span of comprehension!

The second course was a lighter and more varied collation: a visit to the Mathematical Laboratory for a lecture and demonstration by Dr. Wilkes and his co-workers, where EDSAC I behaved well enough to play Noughts and Crosses and EDSAC II was seen under construction with its parts laid open to bewilder the beholder; the time here was all too short: a talk from Dr. Cartwright on her visit to Russia and short talks on Careers for Mathematicians and Changes in the Mathematical Tripos.

Finally, and most valuable of all were the discussions, where the dons told the schoolmasters (in which term twelve schoolmistresses must be included) how little their pupils thought they had to learn, and the schoolmasters told the dons how much they seemed to expect them to teach. These exchanges of views were appreciated by all, and from the late hour to which they invariably continued it may reasonably be concluded that at any future course—and all present hoped there would be many such—more time should be allotted to such informal discussion.

H.M.C.

PROBLEM

The following problem was suggested by Professor A. S. Besicovitch. In how many ways can four points be arranged in a plane so that there are only two values of the distance between any pair of them? (There are six solutions).

The problem can obviously be generalised in many different ways. For example: (i) In how many ways can five points be arranged in a plane under the same conditions? (ii) In how many ways can five points be arranged in three-dimensional space under the same conditions? (iii) In how many ways can six points be arranged in three-dimensional space under these conditions? (iv) What is the maximum number of points p_n that can be arranged under these conditions in n-dimensional space? (v) Most generally of all, what is the maximum number of ways in which p points can be arranged in Euclidean space of n dimensions so that the distances between pairs of points have precisely q different values, expressed in terms of n, p and q.

H.M.C.

LEICESTER AND COUNTY BRANCH

Report for the Session 1955–1956

Seven meetings have been held at Leicester since the last report for this branch was published. They were as follows:

11th October, 1955. Mr. A. Prag of Westminster School gave a talk on "Mathematical Literature", illustrated by books selected from the Mathematical Association Library, now housed at Leicester.

9th November, 1955. Mr. T. R. Theakston of Coventry Training College spoke on "Mathematics in the Modern School—Reality, Reason and Satisfaction".

9th December, 1955. Mr. E. C. Witcombe of Hinckley Grammar School again acted as Question Master at the third annual mathematical quiz between teams from the sixth forms of Grammar Schools in the City and the County. Mr. E. J. F. Primrose of University College Leicester, officiated as Umpire.

28th January, 1956. Mr. W. Hope-Jones spoke on "Some fun with probability".

15th February, 1956. The Annual General Meeting was followed by a talk on "Mathematics as a thinking mechanism" given by Professor R. L. Goodstein of University College, Leicester.

8th March, 1956. Dr. Paul White of Reading University spoke on "What is the matter with negative mass?"

7th May, 1956. Mr. J. C. Boustead, Vice-Principal of the British Railways Staff Training College, Derby, spoke on "Railway Timetables".

The modern schools discussion group, whose formation was described in our last report has grown in size during the session. The group met monthly during the Autumn and Spring terms and had several helpful discussions on problems concerning the teaching of Mathematics in Secondary Modern Schools.

Branch membership has increased slightly. There are now twenty-eight full members and approximately forty associates.

Office holders for the 1956–1957 session are: *President*, Mr. W. Flemming, *Vice-President*, Prof. R. L. Goodstein, *Secretary*, Mr. W. E. Remington, *Treasurer*, Mr. L. G. Clarke.

W. E. Remington (*Hon. Sec.*)

SOUTHAMPTON AND DISTRICT MATHEMATICAL SOCIETY

Report for the Session 1955–1956

The branch has had a very successful year. The number of full members and associate members remain at about 30 and 20 respectively.

Six meetings were held during the year, two in each term. Five of the speakers were as follows:

Dr. Doris M. Lee on "Diagnostic Testing in Mathematics".
Dr. W. L. Ferrar on "Some Thoughts on Text-Books".
Dr. E. A. Maxwell on "On Making Mistakes".
Dr. D. Schonland on "Computing Machines".
Dr. G. N. Lance on "Applications of Mathematics to Aeronautics".

The sixth meeting was devoted to a symposium on "The Teaching of Mechanics". This symposium was very successful (we held a similar one on the teaching of calculus a few years ago). Three topics were discussed, each introduced by a five-minute speech by one of our members, and the subsequent discussions were very interesting.

We have continued our policy of holding our meetings at local grammar schools in turn and it is thought that members prefer this to having them all at the University.

F. G. Maunsell, *Secretary*

THE EXETER MATHEMATICAL SOCIETY

Report for the Session 1955–1956

The Exeter branch was formed at a meeting held on 11th December, 1954, under the auspices of the Department of Education of the then University College of the South West mainly as a result of united efforts by Professor T. Arnold Brown and Mr. A. P. Rollett. The following officers were elected: *President*, Professor T. A. Brown; *Hon. Secretary*, Miss N. A. Comerford; *Hon. Treasurer*, Miss L. G. Button.

The first meeting was held on 29th January, 1955, at which we were lucky enough to persuade Professor T. A. Broadbent to address us on "Why do we teach Mathematics". About 80 people were present, and the branch was off to a very good start.

Further meetings have been as follows:

12th March, 1955	A. P. Rollett, Esq., "Fibonacci Numbers".
29th April, 1955	W. Hope-Jones, Esq., "Fun with Relativity".
22nd June, 1955	A discussion on B-stream Mathematics.
4th November, 1955	Dr. E. A. Maxwell, "Mistakes".
21st March, 1956	Annual General Meeting at which the principal officers were re-elected, followed by a discussion on the shortage of mathematicians.

The present membership is about 70.

C. F. Fisher, *Hon. Secretary*

THE QUEENSLAND MATHEMATICAL SOCIETY

Report for the Session 1955–1956

The Thirty-fourth Annual Report of the Queensland Branch of the Mathematical Association was presented at the annual meeting held on 20th March, 1956; it covers the period May 1955 to March 1956.

The Annual Meeting was held at the University, George Street, on 1st

April, 1955. The Annual Report and the Statement of Receipts and Expenses were presented to the meeting and both were adopted.

This was followed by the election of officers for the coming year.

As a result of a discussion on the question of discussing with teachers of Mathematics the examination papers set in the Public Examinations, it was resolved to recommend to the University Department of Mathematics that any future discussions of this kind be held only under the auspices of the Association.

Associate-Professor J. P. McCarthy, M.A., then read a paper entitled " Quadratic Expressions Associated with Prime Numbers ".

Three General Meetings have been held during the year—all at the University, George Street. At the first, held on 20th May, Mr. I. A. Evans, M.Sc., B.A., read a paper entitled " Axioms of Geometry ". The second meeting, held on 5th August, was devoted to a discussion on " The Teaching of Solid Geometry " ; this proved most interesting. At the third meeting, held on 28th October, Mr. H. M. Finucan, M.A., read a paper entitled " Mathematics and Language ".

The Statement of Receipts and Expenses for the year now ending discloses a credit balance of £11 7s. 6d. Most of the expenses are incurred, as usual, in postal charges—particularly in connection with the circulation of *The Mathematical Gazette* among members.

The number of members at present is 41, which includes 2 Life Members of the Mathematical Association and 13 Ordinary Members of that Association.

The Mathematical Gazette comes to hand regularly and is circulated as expeditiously as circumstances allow, among the members.

The attendance at meetings has been quite satisfactory, and good discussions have followed the reading of the papers at the various meetings.

The Society is very grateful to those members who provide papers and initiate discussions.

MISS M. A. POPPLE, *Hon. Secretary*

BOOKS FOR REVIEW

Aleksandrov, P. S. *Combinatorial Topology, I.* Pp. 225. 1956. $4.95. (Graylock Press, Rochester, N.Y.)

American Mathematical Society. *Proceedings of symposia in applied mathematics, VI. Numerical Analysis.* Pp. 299. 1956. 73s. (McGraw-Hill)

Baggott, E. A. and Thomas, C. *General School Mathematics, II.* Pp. 339 + 32. 1956. 9s. 6d. with answers ; 8s. 6d. without answers. (English Universities Press)

Baldassarri, M. *Algebraic Varieties.* Pp. 195. 1956. DM 36. (Springer, Berlin)

Bass, W. G. and Dowty, O. S. *Counting and Arithmetic in the Infants School.* Pp. 106. 1956. 7s. 6d. (Harrap)

Bennett, A. A., Milne, W. E. and Bateman, H. *Numerical integration of differential equations.* Pp. 108. 1956. $1.35. (Dover Publications, New York)

Bieberbach, L. *Einführung in die theorie der Differentialgleichungen in reellen Gebiet.* Pp. vi + 281. 1956. DM 29.80. (Springer, Berlin)

Blaschke, Muller. *Ebene Kinematik.* Pp. 269. 1956. DM 26.80. (Oldenburg, Munich)

Blaschke, W., Kreis and Kergel. Pp. 164. 1956. DM 18.60. (de Gruyter, Berlin)

Bodewig, E. *Matrix Calculus.* Pp. 334. 1956. (North Holland Publishing Co., Amsterdam)

Booth, A. D. and Booth, K. H. V. *Automatic Digital Calculators.* Pp. 261. 1956. 32s. (Butterworths Scientific Publications)

Born, Max. *Experiment and Theory in Physics.* Pp. 44. 1956. 60c. (Dover Publications, New York)

Bourbaki, N. *Théorie des Ensembles.* Livre I, Chap. III. Pp. 118. 1956. 1,500 frs. (Hermann, Paris)

Briggs, W. and Bryan, G. H. *Tutorial Algebra, II.* Pp. 725. 1956. 24s. (University Tutorial Press)

Carathéodory, C. *Mass und Integral und ihre Algebraisierung.* Pp. 337. 1956. DM 38.50. (Birkhauser, Basel and Stuttgart)

Carathéodory, C. *Variationsrechnung und partielle differentialgleichungen erster Ordnung, I.* Pp. 171. 1956. DM 14. (Teubner, Leipzig)

Carmichael, R. D. *Groups of Finite Order.* Pp. 447. 1956. $2.00. (Dover Publications, New York)

Carson, J. R. *Electric Circuit Theory and Operational Calculus.* Pp. 197. 1956. $1.88. (Chelsea Publishing Co., New York)

Cartan, H. and Eilenberg, S. *Homological Algebra.* Pp. 390. 1956. 45s. (Princeton University Press ; London, Cumberlege)

Centre Belge de Recherches mathématiques. *Colloque sur la Théorie des Nombres.* Pp. 208. 1956. 2,400 frs. (Liège, Georges Thone ; Paris, Masson)

Cesari, L. *Surface Area.* Pp. 594. 1956. 68s. (Princeton University, Press ; London, Cumberlege)

Channon, J. B. and Smith, A. McLeish. *General Mathematics, I.* with answers. Pp. 303. 1956. 7s. 6d. (Longmans)

Chevalley, C. *The Construction and Study of certain important Algebras.* Pp. 64. 1955. (The Mathematical Society of Japan)

Church, A. *Introduction to Mathematical Logic, I.* Pp. 376. 1956. 60s. (Princeton University Press)

Clarke, R. B. *Mathematical Test Papers for Upper Forms.* Pp. 48. 1956. 2s. paper. (University of London Press)

Crank, J. *The Mathematics of Diffusion.* Pp. 347. 1956. 50s. (Oxford, Clarendon Press ; London, Cumberlege)

Daudel, R. *Les Fondements de la Chimie Théorique.* Pp. 236. 1956. 3,500 fr. (Gauthier-Villars)

Destouches, J. L. *La quantification en théorie fonctionelle des corpuscles.* Pp. 141 + vi. 1956. 2,000 fr. (Gauthier-Villars)

Doetsch, G. *Handbuch der Laplace-Transformation, II.* Pp. 433. 1955. (Birkhauser, Basel).

Dollon, J. *Problèmes d'Agrégation.* Pp. 388. 1956. (Libraire Vuibert, Paris)

Douglas, A. V. *Arthur Stanley Eddington:* Pp. 207. 1956. 25s. (Thomas Nelson)

Dowgird, Z. *Krakowiany.* Pp. 168. 1956. (Instytut Podstawowych Problemow Techniki, Warsaw)

Dryden, H. L., Murnaghan, F. P. and Bateman, H. *Hydro-Dynamics.* Pp. 634. 1956. $2.50.

Duff, G. *Partial Differential Equations.* Pp. 248. 1956. 35s. (Toronto, University Press ; London, Cumberlege)

Einstein, A. *The Theory of the Brownian Movement.* Pp. 119. 1956. (Dover Publications, New York)

Erdelyi, A. *Asymptotic Expansions.* Pp. vi + 108. 1956. $1.35. (Dover Publications, New York)

Euclid's Elements—1, 2, 3. Pp. 432. 1956. $5.85 for three volume paper set. (Dover Publications, New York)

Fagg, S. V. *Differential Equations.* Pp. 128. 1956. 7s. 6d. (English Universities Press)

Fleckenstein, J. C. *Der Prioritatsstreit zwischen Leibniz und Newton.* Pp. 27. 1956. (Birkhauser, Basel)

Friedman, B. *Principles and Techniques of Applied Mathematics.* Pp. 313. 64s. 1956. (Wiley, New York ; London ; Chapman and Hall)

Garnier, R. *Cours de Cinématique, II.* Pp. 341. 1956. 70 fr. (Gauthier-Villars. Paris)

Golab, Stanislaw. *Rachunek Tensorowy.* Pp. 309. 1956.

Graded Examples in Elementary Geometry. Pp. 117. 1956. (The Kenion Press)

Grant, H. E. *Practical Descriptive Geometry* (alternate edition). Pp. 403. 1956. 37s. 6d. (McGraw-Hill)

Graves, L. M. *The Theory of Functions of Real Variables.* Pp. 375. 1956. 56s. 6d. (McGraw-Hill)

Grohner, W., Andreotti, A. and Rosati, M. *Abelsche Funktionen und algebraische Geometrie.* Pp. 276. 1956. (Springer, Berlin)

Haack, W. *Elementaire Differentialgeometrie.* Pp. 238. 1955. DM 22. (Birkhauser, Basel)

Hall, D. W. and Spencer, G. L. *Elementary Topology.* Pp. 303. 1955. 56s. (Wiley, New York ; Chapman and Hall)

Hamel, G. *Mechanik der Kontinua.* Pp. 210. 1956. DM 29.70. (Teubner, Stuttgart)

Hannah, J. and Stephens, R. C. *Examples in Mechanical Vibrations.* Pp. 152. 1956. 18s. (Edward Arnold Ltd.)

Heilbronn, G. *Intégration des équations differentielles ordinaires par la méthode de Drach.* Pp. 101. 1956. (Gauthier-Villars, Paris)

Heineman, E. R. *Plane Trigonometry.* Pp. 167. 1956. 24s. 6d. (McGraw-Hill)

Heitler, W. *Elementary Wave Mechanics.* Pp. 190. 1956. 18s. (Oxford, Clarendon Press ; London, Cumberlege)

Henkin, L. *La structure algébrique des théories mathématiques.* Pp. 52. 1956. 900 fr. (Gauthier-Villars, Paris)

Hertz, H. *The Principles of Mechanics presented in a new form.* Pp. 271. 1956. $3.50 cloth bound, $1.75 paper bound. (Dover Publications, New York)

Hildebrand, F. B. *Introduction to Numerical Analysis.* Pp. 511. 1956. 64s. (McGraw-Hill)

Hill, T. H. W. *Problem Tests in Arithmetic.* Pp. 32. 1956. (Harrap)

Hirzebruch, F. *Neue topologische Methoden in der algebraischen Geometrie.* Pp. 165. 1956. DM 30.80. (Springer, Berlin)

Hohenburg, F. *Konstruktive Geometrie für Techniker.* Pp. 272. 1956. (Springer, Berlin)

Information Theory, Third London Symposium 1955. Edited by Colin Cherry. Pp. 401. 1956. 70s. (Butterworths Scientific Publications)

Jacobi, C. G. *Canon Arithmeticus.* Pp. 432. 1956. (Akademie-Verlag, Berlin)

James, E. J. *Modern School Mathematics.* Pp. 159. 1956. 6s. 6d. (Oxford University Press)

Kahan, T., Rideau, G. and Roussopoulos, P. *Les méthodes d'approximation variationnelles dans la théorie des collisions atomiques et dans la physique des piles nucleaires.* Pp. 81. 1956.

Klein, F. *Famous Problems of Elementary Geometry.* Pp. 92. 1956. $1.00. (Dover Publications, New York)

Klein, F. *Lectures on the Icosahedron.* Pp. 289. 1956. $1.85. (Dover Publications, New York)

Kulezycki, S. *Geometria Nieeuklidesowa.* Pp. 188. 1956

Kurosh, A. G. *The Theory of Groups.* Pp. 308. 1956. $4.95. (Chelsea Publishing Co., New York)

LeVeque, W. J. *Topics in Number Theory, I.* Pp. 198. 1956. 44s. (Academic Books)

LeVeque, W. J. *Topics in Number Theory, II.* Pp. 270. 1956. 52s. (Academic Books)

Lukaszewicz, J. and Warmus, M. *Metody Numeryczne I Graficzne.* Pp. 425. 1956

Macintyre, S. and Witte, E. *German-English Mathematical Vocabulary.* Pp. 95. 1956. 8s. 6d. (Oliver and Boyd, Edinburgh)

Macmillan, R. H. *Automation—Friend or Foe.* Pp. 100. 1956. 8s. 6d. (Cambridge University Press)

Martin, W. T. and Reissner, E. *Elementary Differential Equations.* Pp. 260. 1956. $5.50. (Addison-Wesley, Massachusetts)

Mazurkiewicz, S. *Podstawy Rachunku Prawdopodobienstwa.* Pp. 270. 1956. (Monografie Mathematyczne, Warsaw)

McLachlan, N. W. *Ordinary Non-Linear Differential Equations in Engineering and Physical Sciences.* 2nd Edition. Pp. 271. 1956. 35s. (Oxford, Clarendon Press; London, Cumberlege)

Mayer, H. A. *Symposium on Monte Carlo Methods.* Pp. 382. 1956. 60s. (New York, Wiley; London, Chapman and Hall)

Monjallon, A. *Initiation au Calcul Matriciel.* Pp. 127. 1955. 700 fr. (Librairie Vuibert, Paris)

Neville, E. H. *Rectangular-Polar Conversion Tables.* Pp. 109. 1956. 30s. (C.U.P.)

Nichtlineare Regelungsvorgange. Pp. 107. 1956. DM 14. (Oldenbourg, Munich)

Ogilvy, C. S. *Through the Mathescope.* Pp. 162. 1956. 24s. (O.U.P.; London, Cumberlege)

Ostmann, H. H. *Additive Zahlentheorie* Part I. Pp. 233. 1956. DM 29.80. (Springer, Berlin)

Otto, E. *Nomografia.* Pp. 220. 1956.

Palmer, A. H. G. and Snell, K. S. *Mechanics.* Pp. 352. 1956. 25s. (University of London Press)

Patterson, E. M. *Topology.* Pp. 128. 1956. 8s. 6d. (Oliver and Boyd)

Perron, O. *Irrationalzahlen.* Pp. 199. 1956. $1.50. (Chelsea Publishing Co., New York)

Proceedings of the Second International Congress of the International Union for the Philosophy of Science.

I. Exposés généraux
II. Physique-Mathématiques
III. Théorie de la connaissance-Linguistique
IV. Philosophie et Science—Histoire de la philosophie
V. Sociologie-Psychologie.

Pp. 744 complete. 1955. Frs. 38. (Griffon, Switzerland)

Progress in Nuclear Energy. Series I. *Physics and Mathematics, I.* Editors: Charpie, R., Horowitz, J., Hughes, D. J., Litler, D. J. (Pergamon Press Ltd.)

Rajagopal, C. T. and Srinivasaraghaven, V. R. *An Introduction to Analytical Conics.* Pp. 320. 1956. 9s. 6d. (O.U.P.; London, Cumberlege)

Reddick, H. W. and Kibbey, D. E. *Differential Equations.* Pp. 304. 1956. 36s. (Wiley, New York; London, Chapman and Hall)

Riesz, F. and Sz.-Nagy, B. *Functional Analysis.* Pp. xii + 467. 1956. 65s. (Blackie)

Ringleb, H. F. C. *Mathematische Formelsammlung.* Pp. 276 + 15. 1956. DM 4.80. (Walter de Gruyter, Berlin)

Robinson, A. *Complete Theories.* Pp. 129. 1956. (North-Holland Publishing Co., Amsterdam)

Ross Ashby, W. *Cybernetics.* Pp. 295. 1956. 36s. (Chapman and Hall)

Rothe, R. and Schmeidler, W. *Höhere Mathematik, VII.* Pp. 218. 1956. DM 19.80. (Teubner, Stuttgart)

Russell, B. A. W. *Foundations of Geometry.* Pp. xxii + 201. 1956. $1.50; Clothbound $3.25. (Dover Publications, New York)

Scarborough, J. *Numerical Mathematical Analysis,* 3rd Edition. Pp. 554. 1955. 48s. (O.U.P.; London, Cumberlege)

Schmetterer L. *Einführung in die mathematische Statistik.* Pp. xxii + 405. 1956. (Springer, Berlin)

Shannon and McCarthy. *Automata Studies.* Pp. 285. 1956. 32s. (Princeton University Press; London, Cumberlege)

Siegel, C. L. *Vorlesungen über Himmelsmechanik.* Pp. 212. 1956. DM 29.80. (Springer, Berlin)

Smart, W. M. *Foundations of Analytical Geometry.* Pp. 249. 1956. 14s. (Longmans)

Sneddon, I. N. *Special Functions of Mathematical Physics and Chemistry.* Pp. 164. 1956. 10s. 6d. (Oliver and Boyd)

Sokolnikoff, I. S. *Mathematical Theory of Elasticity.* Pp. 476. 1956. 71s. 6d. (McGraw-Hill)

Southwell, R. V. *Relaxation Methods in Theoretical Physics.* Pp. 522. 1956. 55s. (Clarendon Press)

Specht, W. *Gruppentheorie.* Pp. 457. 1956. DM 66. (Springer, Berlin)

Speiser, A. *Die geistige Arbeit.* Pp. 207. 1955. (Birkhauser, Basel)

Spooner, C. R. and Sturgess, P. W. *Exercises in Elementary Geometry.* Pp. 192. 1956. With answers 8s. 6d.; without answers 7s. 6d. (Harrap)

Sticker, H. *How to Calculate Quickly.* Pp. 256. 1956. $1.00. (Dover Publications, New York)

Stoilow, S. *Leçons sur les principes topologiques de la théorie des fonctions analytiques.* Pp. 194. 1956. (Gauthier-Villars, Paris)

Stratton, J. A., Chu, L. J., Morse, P. M., Little, J. D. C., Corbato, F. J. *Spherical Wave Functions.* Pp. 613. 1956. 100s. (John Wiley, New York)

Strubecker, K. *Einführung in die höhere Mathematik, I.* Pp. 820. 1956. DM 36. (Oldenbourg, Munich)

Suzuki, M. *Structure of a Group and the Structure of its Lattice of Subgroups.* Pp. 96. 1956. DM 16.50. (Springer, Berlin)

Swainger, K. *Analysis of Deformation, III. Fluidity.* Pp. 266. 1956. 65s. (Chapman and Hall)

Tables of the Function Arc Sin z. Pp. 586. 1956. 100s. (Harvard University Press; London, Cumberlege)

Tarski, A. *Logic, Semantics, Meta-Mathematics.* Pp. 471. 1956. 60s. (Oxford, Clarendon Press; London, Cumberlege)

Tarski, A. *Ordinal Algebras.* Pp. 133. 1956. 27s. (North-Holland Publishing Company, Amsterdam)

Temple, G. and Bickley, W. G. *Rayleigh's Principle.* Pp. 152. 1956. $1.50. (Dover Publications, New York)

Timoshenko, S. and Young, D. H. *Engineering Mechanics.* Pp. 478 + 51. 1956. 56s. 6d. (McGraw-Hill)

Townsend, A. A. *The Structure of Turbulent Shear Flow.* Pp. 315. 1956. 40s. (Cambridge University Press)

Tucker, J. D. and Wilkinson, D. F. *Radio, III.* Pp. 249. 1956. 12s. 6d. (English Universities Press)

Tuckey, C. O. and Swan, F. J. *Geometry for Sixth Forms.* Pp. 248. 1955. 10s. 6d. (Longmans)

Vajda, S. *The Theory of Games and Linear Programming.* Pp. 106. 1956. 8s. 6d. (Methuen)

Vogel, T. *Physique mathématique classique.* Pp. 214. 1956. (Librairie Armand Colin)

Waerden, B. L. van der, Nievergelt, E. *Tafeln zum Vergleich zweier Stichproben mittels χ-Test und Zeichentest.* Pp. 34. 1956. DM 4.80 (Springer, Berlin)

Ward Hill, T. H. *Elementary Calculations, III.* Pp. 123. 1956. 5s. (Harrap)

Webb, H. *Direct Mathematics.* Books 1, 2, 3, 4 and Teacher's Book. Pp. 119, 136, 135, 136 and 154 respectively. 1956. (Nelson)

Weyl, H. *Selecta.* Pp. 592. 1956. (Birkhauser, Basle and Stuttgart)

Weitzenbock, R. W. *Der vierdimensionale Raum.* Pp. 223. 1956. Fr. DM 19.55. (Birkhauser, Basle and Stuttgart)

Winogradow, I. M. *Elemente der Zahlentheorie.* Pp. 156. 1956. (Oldenbourg, Munich)

Wittich, H. *Neuere Untersuchungen über eindeutige analytische Funktionen.* Pp. 163. 1955. DM 25.60. (Springer, Berlin)

Yiftah, S. *Constantes fondamentales des Théories physiques.* Pp. 122. 1956. (Gauthier-Villars, Paris)

Zygmund, A. *Trigonometrical Series.* Pp. 329. 1955. (Chelsea Publishing Co., New York)

FOR SALE

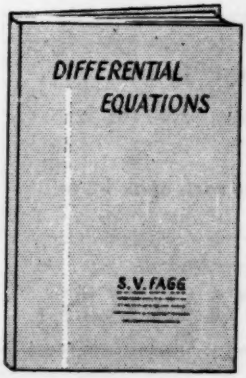
Mathematical Gazette : Nos. 306 to 332 complete. Mr. B. Davies, 8 South Eden Park Road, Beckenham, Kent,

Mathematical Gazette : Nos 288 to 332. D. H. Butler, White Cottage, Snelsmore, Newbury, Berks.

Mathematical Gazette : 1938 to 1955 (September 1947 missing). H. Mary Rickards, Greycot, Dinham, Ludlow, Salop.

WANTED

Will any member who has for disposal a copy of *The Foundations of Euclidean Geometry* by H. G. Forder, Cambridge University Press, please write to P. Kania, Post-Primary Department, Teachers' College, Auckland, S.E. 3, New Zealand, stating the price required, as Auckland Teachers' College is most anxious to obtain this book.



DIFFERENTIAL EQUATIONS

by **S. V. FAGG,** B.Sc., A.R.C.S., D.I.C.,

Assistant Lecturer in Mathematics in the University of Manchester and Lecturer in the Manchester College of Science and Technology.

Inspection copies from; Dept. A44,
THE ENGLISH UNIVERSITIES PRESS
102 Newgate Street, London, E.C.1

"This book could be used by a student during his last year at school or first year at the university, and covers all types of differential equations likely to be encountered in G.C.E. Advanced and Scholarship papers. It should also be particularly useful in technical colleges. Clear worked examples and plenty of well-chosen exercises will be a real help to the student."
The Times Educational Supplement

7/6 net

Also in the Applied Mathematics Series
INTRODUCTION TO ELLIPTIC FUNCTIONS by F. Bowman M.A., M.Sc.Tech.

TO TEACHERS IN MODERN AND TECHNICAL SCHOOLS

Are your pupils being prepared for the needs of modern life?

i *By the contents of the Mathematics Syllabus*
ii *By the methods of instruction*

IF you have any doubts about this send for a free copy of an interesting 64-page booklet (price 1s. 6d.) in which the author, Mr P. F. Burns, explains and justifies the syllabus and method of instructions followed in this new course *Daily Life Mathematics*. As a former H.M. Inspector of Schools Mr Burns knows intimately the mathematical requirements, both practical and cultural of pupils in Modern and Technical Schools, and the four-book course is the outcome of many years' thinking and experimenting.

The books are intended for individual study. The format is large ; the type page open and inviting ; the text friendly, and illustrated by an abundance of diagrams and drawings. Throughout the course visual methods have been used whenever appropriate because they help pupils, particularly the less gifted ones, to grasp more clearly many mathematical processes. These books are well worth looking through and the publishers will be pleased to send loan copies to teachers (postage repaid on books returned).

4 BOOKS IN THIS SERIES *
by P. F. Burns, B.Sc., F.R.A.S.

BOOK ONE
288 pages 10s. 0d.

BOOK TWO
256 pages 10s. 0d.

BOOK THREE
224 pages 10s. 0d.

BOOK FOUR
256 pages 10s. 0d.

Answers to each book 1s. 6d.

*DAILY LIFE MATHEMATICS

Send for a FREE copy of the 64-page booklet

To: GINN AND COMPANY LTD.
18 BEDFORD ROW, LONDON, W.C. 1

Please send a free copy of 64-page booklet *Daily Life Mathematics* and a loan copy of Bks. 1...., 2...., 3...., 4....

(*Tick what is wanted*)

Name .....

School .....

..... :M.G. 71

.....

Scientific Inference

SIR HAROLD JEFFREYS

Since the first edition of this work in 1931 there have been so many developments that Sir Harold has now rewritten the greater part of it. The theory of scientific inference is developed more fully than before, and the applications have been extended. There is an additional chapter on Statistical Mechanics and Quantum Theory, and in a new section a number of widely recommended accounts of scientific procedure are examined and shown to be unworkable as they stand. *Second Edition.* 25*s. net*

Table of the Fresnel Integral

T. PEARCEY

This table of the Fresnel Integral to six decimal places, first published in Australia by the Commonwealth Scientific and Industrial Research Organization, Melbourne, forms a companion volume to *Anti-Composition Tables*, published 1955. 12*s.* 6*d. net*

The Lognormal Distribution

J. AITCHISON & J. A. C. BROWN

The authors present a unified development of the theory of lognormal distribution, with special reference to its use in Economics. A chapter on probit analysis treats this as a development of lognormal estimation theory. The treatment is intended for the practising statistician. 35*s. net*

CAMBRIDGE UNIVERSITY PRESS

BENTLEY HOUSE, 200 EUSTON ROAD, LONDON, N.W.1

TO TEACHERS IN MODERN AND TECHNICAL SCHOOLS

Are your pupils being prepared for the needs of modern life?

i *By the contents of the Mathematics Syllabus*
ii *By the methods of instruction*

IF you have any doubts about this send for a free copy of an interesting 64-page booklet (price 1s. 6d.) in which the author, Mr P. F. Burns, explains and justifies the syllabus and method of instructions followed in this new course *Daily Life Mathematics*. As a former H.M. Inspector of Schools Mr Burns knows intimately the mathematical requirements, both practical and cultural of pupils in Modern and Technical Schools, and the four-book course is the outcome of many years' thinking and experimenting.

The books are intended for individual study. The format is large ; the type page open and inviting ; the text friendly, and illustrated by an abundance of diagrams and drawings. Throughout the course visual methods have been used whenever appropriate because they help pupils, particularly the less gifted ones, to grasp more clearly many mathematical processes. These books are well worth looking through and the publishers will be pleased to send loan copies to teachers (postage repaid on books returned).

4 BOOKS IN THIS SERIES *
by P. F. Burns, B.Sc., F.R.A.S.

BOOK ONE
288 pages 10s. 0d.

BOOK TWO
256 pages 10s. 0d.

BOOK THREE
224 pages 10s. 0d.

BOOK FOUR
256 pages 10s. 0d.

Answers to each book 1s. 6d.

*DAILY LIFE MATHEMATICS

Send for a FREE copy of the 64-page booklet

To: GINN AND COMPANY LTD.
18 BEDFORD ROW, LONDON, W.C. 1

Please send a free copy of 64-page booklet *Daily Life Mathematics* and a loan copy of Bks. 1..., 2..., 3..., 4....

(*Tick what is wanted*)

Name .....

School .....

.....:M.G. 71

Scientific Inference

SIR HAROLD JEFFREYS

Since the first edition of this work in 1931 there have been so many developments that Sir Harold has now rewritten the greater part of it. The theory of scientific inference is developed more fully than before, and the applications have been extended. There is an additional chapter on Statistical Mechanics and Quantum Theory, and in a new section a number of widely recommended accounts of scientific procedure are examined and shown to be unworkable as they stand. *Second Edition.* 25*s. net*

Table of the Fresnel Integral

T. PEARCEY

This table of the Fresnel Integral to six decimal places, first published in Australia by the Commonwealth Scientific and Industrial Research Organization, Melbourne, forms a companion volume to *Anti-Composition Tables*, published 1955. 12*s.* 6*d. net*

The Lognormal Distribution

J. AITCHISON & J. A. C. BROWN

The authors present a unified development of the theory of lognormal distribution, with special reference to its use in Economics. A chapter on probit analysis treats this as a development of lognormal estimation theory. The treatment is intended for the practising statistician. 35*s. net*

CAMBRIDGE UNIVERSITY PRESS

BENTLEY HOUSE, 200 EUSTON ROAD, LONDON, N.W.1

OXFORD BOOKS

SOLID GEOMETRY

By J. S. HAILS, Goldsmiths' College, London, and E. J. HOPKINS, Assistant Master at the Lower School of John Lyon, Harrow.

Pp. 204, with 106 text-figures. Crown 8vo. 13/6 net.

The aim of this book (which is due to appear early in 1957), is to give an account of solid geometry suitable for Sixth Form courses. All the standard properties of the relationships of points, lines, and planes are included. When these have been established the authors discuss polyhedra and then pass on to a treatment of orthogonal projection. There is a full discussion of the cylinder, cone, and sphere and of the right circular cone. A final chapter is devoted to spherical geometry. Throughout the text there are numerous worked examples and plenty of questions for students to answer both of the authors' own devising and drawn from the papers set by examining boards at A level.

By the same authors

AN INTRODUCTION TO PLANE PROJECTIVE GEOMETRY

Pp. 280, with 142 text-figures. Demy 8vo. 27/6 net.

This book has been designed for pupils reading for mathematical scholarships, and first-year undergraduates. The subject, which has been so treated that beginners should be able to read the book with but little assistance, is considered both synthetically and algebraically to ensure a balanced development. The metrical basis of geometry is removed in chapters 1–4, and then the subject is developed non-metrically until the metric gauge is reintroduced as an extension to the argument. The book closes with two chapters on metrical properties of conics and reciprocation.

OXFORD UNIVERSITY PRESS

EDUCATION DEPARTMENT **OXFORD**

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
THE UNIVERSITY PRESS, GLASGOW